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Spinors and field interactions in higher order anisotropic spaces

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Abstract: We formulate the theory of field interactions with higher order anisotropy. The concepts of higher order anisotropic space and locally anisotropic space (in brief, ha–space and la–space) are introduced as general ones for various types of higher order extensions of Lagrange and Finsler geometry and higher dimension (Kaluza–Klein type) spaces. The spinors on ha–spaces are defined in the framework of the geometry of Clifford bundles provided with compatible nonlinear and distinguished connections and metric structures (d–connection and d–metric). The spinor differential geometry of ha–spaces is constructed. There are discussed some related issues connected with the physical aspects of higher order anisotropic interactions for gravitational, gauge, spinor, Dirac spinor and Proca fields. Motion equations in higher order generalizations of Finsler spaces, of the mentioned type of fields, are defined by using bundles of linear and affine frames locally adapted to the nonlinear connection structure.

Keywords: Differential and Algebraic Geometry, Field Theories in Higher Dimensions, Classical Theories of Gravity.

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Contents

1. Introduction

There is a number of fundamental problems in physics advocating the extension to locally anisotropic and higher order anisotropic backgrounds of physical theories [21, 23, 5, 9, 6, 24, 42, 41]. In order to construct physical models on higher order anisotropic spaces it is necessary a corresponding generalization of the spinor theory. Spinor variables and interactions of spinor fields on Finsler spaces were used in a heuristic manner, for instance, in works [6, 25], where the problem of a rigorous definition of la–spinors for la–spaces was not considered. Here we note that, in general, the nontrivial nonlinear connection and torsion structures and possible incompatibility of metric and connections makes the solution of the mentioned problem very sophisticate. The geometric definition of la–spinors and a detailed study of the relationship between Clifford, spinor and nonlinear and distinguished connections structures in vector bundles, generalized Lagrange and Finsler spaces are presented in Refs. [34, 35, 40].

The purpose of this paper is to develop our results [34, 35, 40, 42, 36] on the theory of classical and quantum field interactions on locally anisotropic spaces. Firstly, we receive an attention to the necessary geometric framework, propose an abstract spinor formalism and formulate the differential geometry of higher order anisotropic spaces. The next step is the investigation of higher order anisotropic interactions of fundamental fields on generic higher order anisotropic spaces (in brief we shall use instead of higher order anisotropic the abbreviation ha-, for instance, ha–spaces, ha–interactions and ha–spinors).

In order to develop the higher order anisotropic spinor theory it will be convenient to extend the Penrose and Rindler abstract index formalism [26, 27, 28] (see also the Luehr and Rosenbaum index free methods [20]) proposed for spinors on locally isotropic spaces. We note that in order to formulate the locally anisotropic physics usually we have dimensions $d > 4$ for the fundamental, in general higher order anisotropic space–time, and we must take into account the physical effects of the nonlinear connection structure. In this case the 2-spinor calculus does not play a preferential role.

Section 2 contains an introduction into the geometry of higher order anisotropic spaces, the distinguishing of geometric objects by N–connection structures in such spaces is analyzed, explicit formulas for coefficients of torsions and curvatures of N- and d–connections are presented and the field equations for gravitational interactions with higher order anisotropy are formulated. The distinguished Clifford algebras are introduced in Section 3 and higher order anisotropic Clifford bundles are defined in Section 4. We present a study of almost complex structure for the case of locally anisotropic spaces modeled in the framework of the almost Hermitian model of generalized Lagrange spaces in Section 5. The d–spinor techniques is analyzed in Section 6 and the differential geometry of higher order anisotropic spinors is formulated in Section 7. The Section 8 is devoted to geometric aspects of the theory of field interactions with higher order anisotropy (the d–tensor and d–spinor form of basic field equations

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for gravitational, gauge and d–spinor fields are introduced).

2. Basic geometric objects in ha–spaces

We review some results and methods of the differential geometry of vector bundles provided with nonlinear and distinguished connections and metric structures [22, 23, 24, 41]. This subsection serves the twofold purpose of establishing of abstract index denotations and starting the geometric backgrounds which are used in the next subsections of the section.

2.1. N-connections and distinguishing of geometric objects

Let $\mathcal{E}^{<z>}= (E^{<z>}, p, M, Gr, F^{<z>})$ be a locally trivial distinguished vector bundle, dv-bundle, where $F^{<z>} = \mathcal{R}^{m_1} \oplus \cdots \oplus \mathcal{R}^{m_z}$ (a real vector space of dimension $m = m_1 + \cdots + m_z, \dim F = m$, $\mathcal{R}$ denotes the real number field) is the typical fibre, the structural group is chosen to be the group of automorphisms of $\mathcal{R}^m$, i.e. $Gr = GL(m, \mathcal{R})$, and $p : E^{<z>} \to M$ (defined by intermediary projections $p_{<z,z-1>} : E^{<z>} \to E^{<z-1>}, p_{<z-1,z-2>} : E^{<z-1>} \to E^{<z-2>}, \ldots p : E^{<1>} \to M$) is a differentiable surjection of a differentiable manifold E (total space, $\dim E = n + m$) to a differentiable manifold M (base space, $\dim M = n$). Local coordinates on $\mathcal{E}^{<z>}$ are denoted as

$$u^{<\alpha>} = \left(x^{\mathbf{i}}, y^{<\mathbf{a}>}\right) = \left(x^{\mathbf{i}} \doteq y^{\mathbf{a}_0}, y^{\mathbf{a}_1}, \ldots, y^{\mathbf{a}_z}\right) =$$

$$(\ldots, y^{\mathbf{a}_{(p)}}, \ldots) = \{y_{(p)}^{\mathbf{a}_{(p)}}\} = \{y^{\mathbf{a}_{(p)}}\},$$

or in brief $\mathbf{u} = \mathbf{u}_{<\mathbf{z}>} = (\mathbf{x}, \mathbf{y}_{(1)}, \ldots, \mathbf{y}_{(p)}, \ldots, \mathbf{y}_{(z)})$ where boldfaced indices will be considered as coordinate ones for which the Einstein summation rule holds (Latin indices $\mathbf{i}, \mathbf{j}, \mathbf{k}, \ldots = \mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0, \ldots = 1, 2, \ldots, n$ will parametrize coordinates of geometrical objects with respect to a base space M, Latin indices $\mathbf{a}_p, \mathbf{b}_p, \mathbf{c}_p, \ldots = 1, 2, \ldots, m_{(p)}$ will parametrize fibre coordinates of geometrical objects and Greek indices $\alpha, \beta, \gamma, \ldots$ are considered as cumulative ones for coordinates of objects defined on the total space of a v-bundle). We shall correspondingly use abstract indices $\alpha = (i, a)$, $\beta = (j, b), \gamma = (k, c), \ldots$ in the Penrose manner [26, 27, 28] in order to mark geometrical objects and theirs (base, fibre)-components or, if it will be convenient, we shall consider boldfaced letters (mainly for pointing to the operator character of tensors and spinors into consideration) of type $\mathbf{A} \equiv A = \left(A^{(h)}, A^{(v_1)}, \ldots, A^{(v_z)}\right), \mathbf{b} = \left(b^{(h)}, b^{(v_1)}, \ldots, b^{(v_z)}\right), \ldots, \mathbf{R}, \omega$, $\mathbf{\Gamma}, \ldots$ for geometrical objects on $\mathcal{E}$ and theirs splitting into horizontal (h), or base, and vertical (v), or fibre, components. For simplicity, we shall prefer writing out of abstract indices instead of boldface ones if this will not give rise to ambiguities.

Coordinate transforms $u^{<\alpha'>} = u^{<\alpha'>}(u^{<\alpha>})$ on $\mathcal{E}^{<z>}$ are written as

$$\{u^{<\alpha>} = \left(x^{\mathbf{i}}, y^{<\mathbf{a}>}\right)\} \to \{u^{<\alpha'>} = \left(x^{\mathbf{i}'}, y^{<\mathbf{a}'>}\right)\}$$

and written as recurrent maps

$$x^{\mathbf{i}'} = x^{\mathbf{i}'}(x^{\mathbf{i}}),\ rank\left(\frac{\partial x^{\mathbf{i}'}}{\partial x^{\mathbf{i}}}\right) = n, \tag{1}$$

$$y_{(1)}^{\mathbf{a}_1'} = K_{\mathbf{a}_1}^{\mathbf{a}_1'}(x^{\mathbf{i}})y_{(1)}^{\mathbf{a}_1}, K_{\mathbf{a}_1}^{\mathbf{a}_1'}(x^{\mathbf{i}}) \in GL(m_1, \mathcal{R}),$$

$$\ldots\ldots\ldots\ldots$$

$$y_{(p)}^{\mathbf{a}_p'} = K_{\mathbf{a}_p}^{\mathbf{a}_p'}(u_{(p-1)})y_{(p)}^{\mathbf{a}_p}, K_{\mathbf{a}_p}^{\mathbf{a}_p'}(u_{(p-1)}) \in GL(m_p, \mathcal{R}),$$

$$\ldots\ldots\ldots\ldots$$

$$y_{(z)}^{\mathbf{a}_z'} = K_{\mathbf{a}_z}^{\mathbf{a}_z'}(u_{(z-1)})y_{(z-1)}^{\mathbf{a}_z}, K_{\mathbf{a}_z}^{\mathbf{a}_z'}(u_{(z-1)}) \in GL(m_z, \mathcal{R})$$

where matrices $K_{\mathbf{a}_1}^{\mathbf{a}_1'}(x^{\mathbf{i}}), \ldots, K_{\mathbf{a}_p}^{\mathbf{a}_p'}(u_{(p-1)}), \ldots, K_{\mathbf{a}_z}^{\mathbf{a}_z'}(u_{(z-1)})$ are functions of necessary smoothness class. In brief we write transforms (1) in the form

$$x^{\mathbf{i}'} = x^{\mathbf{i}'}(x^{\mathbf{i}}), y^{<\mathbf{a}'>} = K_{<\mathbf{a}>}^{<\mathbf{a}'>}y^{<\mathbf{a}>}.$$

In general form we shall write K–matrices $K_{<\alpha>}^{<\alpha'>} = \left(K_{\mathbf{i}}^{\mathbf{i}'}, K_{<\mathbf{a}>}^{<\mathbf{a}'>}\right)$, where $K_{\mathbf{i}}^{\mathbf{i}'} = \frac{\partial x^{\mathbf{i}'}}{\partial x^{\mathbf{i}}}$.

A local coordinate parametrization of $\mathcal{E}^{<z>}$ naturally defines a coordinate basis of the module of d–vector fields $\Xi(\mathcal{E}^{<z>})$,

$$\partial_{<\alpha>} = (\partial_i, \partial_{<a>}) = (\partial_i, \partial_{a_1}, \ldots, \partial_{a_p}, \ldots, \partial_{a_z}) = \tag{2}$$

$$\frac{\partial}{\partial u^{<\alpha>}} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{<a>}}\right) = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{a_1}}, \ldots \frac{\partial}{\partial y^{a_p}}, \ldots, \frac{\partial}{\partial y^{a_z}}\right),$$

and the reciprocal to (2) coordinate basis

$$d^{<\alpha>} = (d^i, d^{<a>}) = (d^i, d^{a_1}, \ldots, d^{a_p}, \ldots, d^{a_z}) = \tag{3}$$

$$du^{<\alpha>} = (dx^i, dy^{<a>}) = (dx^i, dy^{a_1}, \ldots, dy^{a_p}, \ldots, dy^{a_z}),$$

which is uniquely defined from the equations

$$d^{<\alpha>} \circ \partial_{<\beta>} = \delta_{<\beta>}^{<\alpha>},$$

where $\delta_{<\beta>}^{<\alpha>}$ is the Kronecher symbol and by "$\circ$" we denote the inner (scalar) product in the tangent bundle $\mathcal{TE}^{<z>}$.

The concept of **nonlinear connection,** in brief, N–connection, is fundamental in the geometry of locally anisotropic and higher order anisotropic spaces (see a detailed study and basic references in [22, 23, 24]). In a dv–bundle $\mathcal{E}^{<z>}$ it is defined as a distribution $\{N : E_u \to H_uE, T_uE = H_uE \oplus V_u^{(1)}E \oplus \cdots \oplus V_u^{(p)}E \cdots \oplus V_u^{(z)}E\}$ on $E^{<z>}$ being a global decomposition, as a Whitney sum, into horizontal, $\mathcal{HE}$, and vertical, $\mathcal{VE}^{<p>}$, $p = 1, 2, \ldots, z$ subbundles of the tangent bundle $\mathcal{TE}$:

$$\mathcal{TE} = H\mathcal{E} \oplus V\mathcal{E}^{<1>} \oplus \cdots \oplus V\mathcal{E}^{<p>} \oplus \cdots \oplus V\mathcal{E}^{<z>}. \tag{4}$$

Locally a N-connection in $\mathcal{E}^{<z>}$ is given by it components $N^{<\mathbf{a}_p>}_{<\mathbf{a}_f>}(\mathbf{u}), z \geq p > f \geq 0$ (in brief we shall write $N^{<a_p>}_{<a_f>}(u)$) with respect to bases (2) and (3)):

$$\mathbf{N} = N^{<a_p>}_{<a_f>}(u)\delta^{<a_f>} \otimes \delta_{<a_p>}, (z \geq p > f \geq 0),$$

We note that a linear connection in a dv-bundle $\mathcal{E}^{<z>}$ can be considered as a particular case of a N-connection when $N_i^{<a>}(u) = K^{<a>}_{<b>i}(x)\, y^{<b>}$, where functions $K^{<b>}_{<a>i}(x)$ on the base M are called the Christoffel coefficients.

To coordinate locally geometric constructions with the global splitting of $\mathcal{E}^{<z>}$ defined by a N-connection structure, we have to introduce a locally adapted basis (la–basis, la–frame):

$$\delta_{<\alpha>} = (\delta_i, \delta_{<a>}) = (\delta_i, \delta_{a_1}, \ldots, \delta_{a_p}, \ldots, \delta_{a_z}), \tag{5}$$

with components parametrized as

$$\delta_i = \partial_i - N_i^{a_1}\partial_{a_1} - \cdots - N_i^{a_z}\partial_{a_z},$$

$$\delta_{a_1} = \partial_{a_1} - N_{a_1}^{a_2}\partial_{a_2} - \cdots - N_{a_1}^{a_z}\partial_{a_z},$$

$$\ldots\ldots\ldots$$

$$\delta_{a_p} = \partial_{a_p} - N_{a_p}^{a_{p+1}}\partial_{a_{p+1}} - \cdots - N_{a_p}^{a_z}\partial_{a_z},$$

$$\ldots\ldots\ldots$$

$$\delta_{a_z} = \partial_{a_z}$$

and it dual la–basis

$$\delta^{<\alpha>} = (\delta^i, \delta^{<a>}) = \left(\delta^i, \delta^{a_1}, \ldots, \delta^{a_p}, \ldots, \delta^{a_z}\right), \tag{6}$$

$$\delta x^i = dx^i,$$

$$\delta y^{a_1} = dy^{a_1} + M_i^{a_1} dx^i,$$

$$\delta y^{a_2} = dy^{a_2} + M_{a_1}^{a_2} dy^{a_1} + M_i^{a_2} dx^i,$$

$$\ldots\ldots\ldots$$

$$\delta y^{a_p} = dy^{a_p} + M_{a_{p-1}}^{a_p} dy^{p-1} + M_{a_{p-2}}^{a_p} dy^{a_{p-2}} + \cdots + M_i^{a_p} dx^i,$$

$$\ldots\ldots\ldots$$

$$\delta y^{a_z} = dy^{a_z} + M_{a_{z-1}}^{a_z} dy^{z-1} + M_{a_{z-2}}^{a_z} dy^{a_{z-2}} + \cdots + M_i^{a_z} dx^i.$$

The **nonholonomic coefficients** $\mathbf{w} = \{w^{<\alpha>}_{<\beta><\gamma>}(u)\}$ of the locally adapted to the N–connection structure frames are defined as

$$[\delta_{<\alpha>}, \delta_{<\beta>}] = \delta_{<\alpha>}\delta_{<\beta>} - \delta_{<\beta>}\delta_{<\alpha>} = w^{<\alpha>}_{<\beta><\gamma>}(u)\, \delta_{<\alpha>}.$$

The **algebra of tensorial distinguished fields** $DT\left(\mathcal{E}^{<z>}\right)$ (d–fields, d–tensors, d–objects) on $\mathcal{E}^{<z>}$ is introduced as the tensor algebra
$\mathcal{T}=\{\mathcal{T}_{qs_1\cdots s_p\cdots s_z}^{pr_1\cdots r_p\cdots r_z}\}$ of the dv-bundle $\mathcal{E}_{(d)}^{<z>}$,

$$p_d : \mathcal{H}\mathcal{E}^{<z>}\oplus\mathcal{V}^1\mathcal{E}^{<z>}\oplus\cdots\oplus V^p\mathcal{E}^{<z>}\oplus\cdots\oplus V^z\mathcal{E}^{<z>}\rightarrow\mathcal{E}^{<z>},$$

An element $\mathbf{t}\in\mathcal{T}_{qs_1\cdots s_z}^{pr_1\cdots r_z}$, d-tensor field of type $\begin{pmatrix} p & r_1 & \cdots & r_p & \cdots & r_z \\ q & s_1 & \cdots & s_p & \cdots & s_z \end{pmatrix}$, can be written in local form as

$$\mathbf{t}=t_{j_1\cdots j_q b_1^{(1)}\cdots b_{r_1}^{(1)}\cdots b_1^{(p)}\cdots b_{r_p}^{(p)}\cdots b_1^{(z)}\cdots b_{r_z}^{(z)}}^{i_1\cdots i_p a_1^{(1)}\cdots a_{r_1}^{(1)}\cdots a_1^{(p)}\cdots a_{r_p}^{(p)}\cdots a_1^{(z)}\cdots a_{r_z}^{(z)}}\left(u\right)\delta_{i_1}\otimes\cdots\otimes\delta_{i_p}\otimes d^{j_1}\otimes\cdots\otimes d^{j_q}\otimes$$

$$\delta_{a_1^{(1)}}\otimes\cdots\otimes\delta_{a_{r_1}^{(1)}}\otimes\delta^{b_1^{(1)}}\cdots\otimes\delta^{b_{s_1}^{(1)}}\otimes\cdots\otimes\delta_{a_1^{(p)}}\otimes\cdots\otimes\delta_{a_{r_p}^{(p)}}\otimes\cdots\otimes$$

$$\delta^{b_1^{(p)}}\cdots\otimes\delta^{b_{s_p}^{(p)}}\otimes\delta_{a_1^{(z)}}\otimes\cdots\otimes\delta_{a_{r_z}^{(z)}}\otimes\delta^{b_1^{(z)}}\cdots\otimes\delta^{b_{s_z}^{(z)}}.$$

We shall respectively use denotations $X(\mathcal{E}^{<z>})$ (or $X(M)$), $\Lambda^p\left(\mathcal{E}^{<z>}\right)$ (or $\Lambda^p\left(M\right)$) and $\mathcal{F}(\mathcal{E}^{<z>})$ (or $\mathcal{F}\left(M\right)$) for the module of d-vector fields on $\mathcal{E}^{<z>}$ (or M), the exterior algebra of p-forms on $\mathcal{E}^{<z>}$ (or M) and the set of real functions on $\mathcal{E}^{<z>}$(or M).

In general, d–objects on $\mathcal{E}^{<z>}$ are introduced as geometric objects with various group and coordinate transforms coordinated with the N–connection structure on $\mathcal{E}^{<z>}$. For example, a d–connection D on $\mathcal{E}^{<z>}$ is defined as a linear connection D on $E^{<z>}$ conserving under a parallelism the global decomposition (4) into horizontal and vertical subbundles of $\mathcal{T}\mathcal{E}^{<z>}$.

A N-connection in $\mathcal{E}^{<z>}$ induces a corresponding decomposition of d-tensors into sums of horizontal and vertical parts, for example, for every d-vector $X\in\mathcal{X}(\mathcal{E}^{<z>})$ and 1-form $\widetilde{X}\in\Lambda^1\left(\mathcal{E}^{<z>}\right)$ we have respectively

$$X=hX+v_1X+\cdots+v_zX \quad \text{and} \quad \widetilde{X}=h\widetilde{X}+v_1\widetilde{X}+\cdots\ v_z\widetilde{X}. \qquad (7)$$

In consequence, we can associate to every d-covariant derivation along the d-vector (7), $D_X=X\circ D$, two new operators of h- and v-covariant derivations defined respectively as

$$D_X^{(h)}Y=D_{hX}Y$$

and

$$D_X^{(v_1)}Y=D_{v_1X}Y,\ldots,\mathbf{D_X^{(v_z)}Y=D_{v_zX}Y}\quad\forall Y\in\mathcal{X}(\mathcal{E}^{<z>}),$$

for which the following conditions hold:

$$D_XY=D_X^{(h)}Y+D_X^{(v_1)}Y+\cdots+D_X^{(v_z)}Y, \qquad (8)$$

$$D_X^{(h)}f=(hX)f$$

and

$$D_X^{(v_p)}f=(v_pX)f,\quad X,Y\in\mathcal{X}\left(\mathcal{E}\right),f\in\mathcal{F}\left(M\right),p=1,2,\ldots z.$$

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We define a **metric structure** $\mathbf{G}$ in the total space $E^{<z>}$ of dv-bundle $\mathcal{E}^{<z>}=(E^{<z>},p,M)$ over a connected and paracompact base M as a symmetrical covariant tensor field of type $(0,2)$, $G_{<\alpha><\beta>}$, being nondegenerate and of constant signature on $E^{<z>}$.

Nonlinear connection $\mathbf{N}$ and metric $\mathbf{G}$ structures on $\mathcal{E}^{<z>}$ are mutually compatible it there are satisfied the conditions:

$$\mathbf{G}\left(\delta_{a_f},\delta_{a_p}\right)=0, \text{or equivalently, } G_{a_f a_p}(u)-N^{<b>}_{a_f}(u)\,h_{a_f<b>}(u)=0, \tag{9}$$

where $h_{a_p b_p}=\mathbf{G}\left(\partial_{a_p},\partial_{b_p}\right)$ and $G_{b_f a_p}=\mathbf{G}\left(\partial_{b_f},\partial_{a_p}\right), 0\leq f<p\leq z,$ which gives

$$N^{b_p}_{c_f}(u)=h^{<a>b_p}(u)\,G_{c_f<a>}(u) \tag{10}$$

(the matrix $h^{a_p b_p}$ is inverse to $h_{a_p b_p}$). In consequence one obtains the following decomposition of metric :

$$\mathbf{G}(X,Y)=\mathbf{hG}(X,Y)+\mathbf{v}_1\mathbf{G}(X,Y)+\cdots+\mathbf{v}_z\mathbf{G}(X,Y), \tag{11}$$

where the d-tensor $\mathbf{hG}(X,Y)=\mathbf{G}(hX,hY)$ is of type $\begin{pmatrix}0\ 0\\ 2\ 0\end{pmatrix}$ and the d-tensor $\mathbf{v}_p\mathbf{G}(X,Y)=\mathbf{G}(v_pX,v_pY)$ is of type $\begin{pmatrix}0 & \cdots & 0(p) & \cdots & 0\\ 0 & \cdots & 2 & \cdots & z\end{pmatrix}$. With respect to la–basis (6) the d–metric (11) is written as

$$\mathbf{G}=g_{<\alpha><\beta>}(u)\,\delta^{<\alpha>}\otimes\delta^{<\beta>}=g_{ij}(u)\,d^i\otimes d^j+h_{<a><b>}(u)\,\delta^{<a>}\otimes\delta^{<b>}, \tag{12}$$

where $g_{ij}=\mathbf{G}\left(\delta_i,\delta_j\right).$

A metric structure of type (11) (equivalently, of type (12)) or a metric on $E^{<z>}$ with components satisfying constraints (9), equivalently (10)) defines an adapted to the given N-connection inner (d–scalar) product on the tangent bundle $\mathcal{TE}^{<z>}$.

We shall say that a d-connection $\widehat{D}_X$ is compatible with the d-scalar product on $\mathcal{TE}^{<z>}$ (i.e. is a standard d-connection) if

$$\widehat{D}_X(\mathbf{X}\cdot\mathbf{Y})=\left(\widehat{D}_X\mathbf{Y}\right)\cdot\mathbf{Z}+\mathbf{Y}\cdot\left(\widehat{D}_X\mathbf{Z}\right),\forall\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathcal{X}(\mathcal{E}^{<z>}).$$

An arbitrary d–connection D_X differs from the standard one $\widehat{D}_X$ by an operator $\widehat{P}_X(u)=\{X^{<\alpha>}\widehat{P}^{<\gamma>}_{<\alpha><\beta>}(u)\}$, called the deformation d-tensor with respect to $\widehat{D}_X$, which is just a d-linear transform of $\mathcal{E}^{<z>}_u$, $\forall u\in\mathcal{E}^{<z>}$. The explicit form of $\widehat{P}_X$ can be found by using the corresponding axiom defining linear connections [20]

$$\left(D_X-\widehat{D}_X\right)fZ=f\left(D_X-\widehat{D}_X\right)Z,$$

written with respect to la-bases (5) and (6). From the last expression we obtain

$$\widehat{P}_X(u)=\left[(D_X-\widehat{D}_X)\delta_{<\alpha>}(u)\right]\delta^{<\alpha>}(u),$$

therefore

$$D_X Z \ = \widehat{D}_X Z + \widehat{P}_X Z. \tag{13}$$

A d-connection D_X is **metric** (or compatible with metric $\mathbf{G}$) on $\mathcal{E}^{<z>}$ if

$$D_X \mathbf{G} = 0, \forall X \in \mathcal{X}(\mathcal{E}^{<z>}). \tag{14}$$

Locally adapted components $\Gamma^{<\alpha>}_{<\beta><\gamma>}$ of a d-connection $D_{<\alpha>} = (\delta_{<\alpha>} \circ D)$ are defined by the equations

$$D_{<\alpha>}\delta_{<\beta>} = \Gamma^{<\gamma>}_{<\alpha><\beta>}\delta_{<\gamma>},$$

from which one immediately follows

$$\Gamma^{<\gamma>}_{<\alpha><\beta>}(u) = (D_{<\alpha>}\delta_{<\beta>}) \circ \delta^{<\gamma>}. \tag{15}$$

The operations of h- and v$_{(p)}$-covariant derivations, $D_k^{(h)} = \{L^i_{jk}, L^{<a>}_{<b>k}\}$ and $D^{(v_p)}_{c_p} = \{C^i_{jc_p}, C^i_{<b>c_p}, C^{<a>}_{jc_p}, C^{<a>}_{<b>c_p}\}$ (see (8)), are introduced as corresponding h- and v$_{(p)}$-parametrizations of (15):

$$L^i_{jk} = (D_k\delta_j) \circ d^i, \quad L^{<a>}_{<b>k} = (D_k\delta_{<b>}) \circ \delta^{<a>} \tag{16}$$

and

$$C^i_{jc_p} = \left(D_{c_p}\delta_j\right) \circ \delta^i, \quad C^{<a>}_{<b>c_p} = \left(D_{c_p}\delta_{<b>}\right) \circ \delta^{<a>} \tag{17}$$

$$C^i_{<b>c_p} = \left(D_{c_p}\delta_{<b>}\right) \circ \delta^i, \quad C^{<a>}_{jc_p} = \left(D_{c_p}\delta_j\right) \circ \delta^{<a>}.$$

A set of components (16) and (17), $D\Gamma = \left(L^i_{jk}, L^{<a>}_{<b>k}, C^i_{j<c>}, C^{<a>}_{<b><c>}\right)$, completely defines the local action of a d-connection D in $\mathcal{E}^{<z>}$. For instance, taken a d-tensor field of type $\begin{pmatrix} 1 & \cdots & 1(p) & \cdots \\ 1 & \cdots & 1(p) & \cdots \end{pmatrix}$, $\mathbf{t} = t^{ia_p}_{jb_p}\delta_i \otimes \delta_{a_p} \otimes \delta^j \otimes \delta^{b_p}$, and a d-vector $\mathbf{X} = X^i\delta_i + X^{<a>}\delta_{<a>}$ we have

$$D_X\mathbf{t} = D_X^{(h)}\mathbf{t} + D_X^{(v_1)}\mathbf{t} + \cdots + \mathbf{D}_{\mathbf{X}}^{(\mathbf{v_p})}\mathbf{t} + \cdots + \mathbf{D}_{\mathbf{X}}^{(\mathbf{v_z})}\mathbf{t} =$$
$$\left(X^k t^{ia_p}_{jb_p|k} + X^{<c>} t^{ia_p}_{jb_p \perp <c>}\right) \delta_i \otimes \delta_{a_p} \otimes d^j \otimes \delta^{b_p},$$

where the h–covariant derivative is written as

$$t^{ia_p}_{jb_p|k} = \frac{\delta t^{ia_p}_{jb_p}}{\delta x^k} + L^i_{hk} t^{ha_p}_{jb_p} + L^{a_p}_{c_p k} t^{ic_p}_{jb_p} - L^h_{jk} t^{ia_p}_{hb_p} - L^{c_p}_{b_p k} t^{ia_p}_{jc_p}$$

and the v–covariant derivatives are written as

$$t^{ia_p}_{jb_p \perp <c>} = \frac{\partial t^{ia_p}_{jb_p}}{\partial y^{<c>}} + C^i_{h<c>} t^{ha_p}_{jb_p} + C^{a_p}_{d_p<c>} t^{id_p}_{jb_p} - C^h_{j<c>} t^{ia_p}_{hb_p} - C^{d_p}_{b_p<c>} t^{ia_p}_{jd_p}.$$

For a scalar function $f \in \mathcal{F}(\mathcal{E}^{<z>})$ we have

$$D_i^{(h)} = \frac{\delta f}{\delta x^i} = \frac{\partial f}{\partial x^i} - N_i^{<a>}\frac{\partial f}{\partial y^{<a>}},$$

$$D_{a_f}^{(v_f)} = \frac{\delta f}{\delta x^{a_f}} = \frac{\partial f}{\partial x^{a_f}} - N_{a_f}^{a_p} \frac{\partial f}{\partial y^{a_p}}, 1 \leq f < p \leq z-1,$$

$$\text{and } D_{c_z}^{(v_z)} f = \frac{\partial f}{\partial y^{c_z}}.$$

We emphasize that the geometry of connections in a dv-bundle $\mathcal{E}^{<z>}$ is very reach. If a triple of fundamental geometric objects

$$\left(N_{a_f}^{a_p}(u), \Gamma_{<\beta><\gamma>}^{<\alpha>}(u), G_{<\alpha><\beta>}(u)\right)$$

is fixed on $\mathcal{E}^{<z>}$, a multiconnection structure (with corresponding different rules of covariant derivation, which are, or not, mutually compatible and with the same, or not, induced d-scalar products in $\mathcal{TE}^{<z>}$) is defined on this dv-bundle. For instance, we enumerate some of connections and covariant derivations which can present interest in investigation of locally anisotropic gravitational and matter field interactions:

1. Every N-connection in $\mathcal{E}^{<z>}$, with coefficients $N_{a_f}^{a_p}(u)$ being differentiable on y–variables, induces a structure of linear connection $\widetilde{N}_{<\beta><\gamma>}^{<\alpha>}$, where $\widetilde{N}_{b_p c_f}^{a_p} = \frac{\partial N_{c_f}^{a_p}}{\partial y^{b_p}}$ and $\widetilde{N}_{b_p c_p}^{a_p}(u) = 0$. For some

$$Y(u) = Y^i(u)\partial_i + Y^{<a>}(u)\partial_{<a>}$$

and

$$B(u) = B^{<a>}(u)\partial_{<a>}$$

one writes

$$D_Y^{(\widetilde{N})} B = \left[Y^{c_f} \left(\frac{\partial B^{a_p}}{\partial y^{c_f}} + \widetilde{N}_{b_p i}^{a_p} B^{b_p} \right) + Y^{b_p} \frac{\partial B^{a_p}}{\partial y^{b_p}} \right] \frac{\partial}{\partial y^{a_p}} \ (0 \leq f < p \leq z).$$

2. The d–connection of Berwald type [10]

$$\Gamma_{<\beta><\gamma>}^{(B)<\alpha>} = \left(L_{jk}^i, \frac{\partial N_k^{<a>}}{\partial y^{<b>}}, 0, C_{<b><c>}^{<a>} \right), \tag{18}$$

where

$$L_{.jk}^i(u) = \frac{1}{2} g^{ir} \left(\frac{\delta g_{jk}}{\partial x^k} + \frac{\delta g_{kr}}{\partial x^j} - \frac{\delta g_{jk}}{\partial x^r} \right), \tag{19}$$

$$C_{.<b><c>}^{<a>}(u) = \frac{1}{2} h^{<a><d>} \left(\frac{\delta h_{<b><d>}}{\partial y^{<c>}} + \frac{\delta h_{<c><d>}}{\partial y^{<b>}} - \frac{\delta h_{<b><c>}}{\partial y^{<d>}} \right),$$

which is hv-metric, i.e. $D_k^{(B)} g_{ij} = 0$ and $D_{<c>}^{(B)} h_{<a><b>} = 0$.

3. The canonical d–connection $\mathbf{\Gamma^{(c)}}$ associated to a metric $\mathbf{G}$ of type (12) $\Gamma^{(c)<\alpha>}_{<\beta><\gamma>} = \left(L^{(c)i}_{jk}, L^{(c)<a>}_{<b>k}, C^{(c)i}_{j<c>}, C^{(c)<a>}_{<b><c>}\right)$, with coefficients

$$L^{(c)i}_{jk} = L^{i}_{.jk}, C^{(c)<a>}_{<b><c>} = C^{<a>}_{.<b><c>} \text{ (see (19))}$$

$$L^{(c)<a>}_{<b>i} = \widetilde{N}^{<a>}_{<b>i} + \tag{20}$$

$$\frac{1}{2}h^{<a><c>}\left(\frac{\delta h_{<b><c>}}{\delta x^{i}} - \widetilde{N}^{<d>}_{<b>i}h_{<d><c>} - \widetilde{N}^{<d>}_{<c>i}h_{<d><b>}\right),$$

$$C^{(c)i}_{j<c>} = \frac{1}{2}g^{ik}\frac{\partial g_{jk}}{\partial y^{<c>}}.$$

This is a metric d–connection which satisfies conditions

$$D^{(c)}_{k}g_{ij} = 0, D^{(c)}_{<c>}g_{ij} = 0, D^{(c)}_{k}h_{<a><b>} = 0, D^{(c)}_{<c>}h_{<a><b>} = 0.$$

4. We can consider N-adapted Christoffel d–symbols

$$\widetilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} =$$

$$\frac{1}{2}G^{<\alpha><\tau>}\left(\delta_{<\gamma>}G_{<\tau><\beta>} + \delta_{<\beta>}G_{<\tau><\gamma>} - \delta_{<\tau>}G_{<\beta><\gamma>}\right), \tag{21}$$

which have the components of d-connection

$$\widetilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} = \left(L^{i}_{jk}, 0, 0, C^{<a>}_{<b><c>}\right)$$

with L^{i}_{jk} and $C^{<a>}_{<b><c>}$ as in (19) if $G_{<\alpha><\beta>}$ is taken in the form (12).

Arbitrary linear connections on a dv–bundle $\mathcal{E}^{<z>}$ can be also characterized by theirs deformation tensors (see (13)) with respect, for instance, to d–connection (21):

$$\Gamma^{(B)<\alpha>}_{<\beta><\gamma>} = \widetilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} + P^{(B)<\alpha>}_{<\beta><\gamma>}, \Gamma^{(c)<\alpha>}_{<\beta><\gamma>} = \widetilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} + P^{(c)<\alpha>}_{<\beta><\gamma>}$$

or, in general,

$$\Gamma^{<\alpha>}_{<\beta><\gamma>} = \widetilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} + P^{<\alpha>}_{<\beta><\gamma>},$$

where $P^{(B)<\alpha>}_{<\beta><\gamma>}, P^{(c)<\alpha>}_{<\beta><\gamma>}$ and $P^{<\alpha>}_{<\beta><\gamma>}$ are respectively the deformation d–tensors of d-connections (18), (20), or of a general one.

2.2. Torsions and curvatures of N- and d-connections

The curvature $\mathbf{\Omega}$ of a nonlinear connection $\mathbf{N}$ in a dv-bundle $\mathcal{E}^{<z>}$ can be defined as the Nijenhuis tensor field $N_{v}(X,Y)$ associated to $\mathbf{N}$ [22, 23]:

$$\mathbf{\Omega} = N_{v} = [\mathbf{vX}, \mathbf{vY}] + \mathbf{v}[\mathbf{X}, \mathbf{Y}] - \mathbf{v}[\mathbf{vX}, \mathbf{Y}] - \mathbf{v}[\mathbf{X}, \mathbf{vY}], \mathbf{X}, \mathbf{Y} \in \mathcal{X}(\mathcal{E}^{<z>}),$$

where $v = v_1 \oplus \cdots \oplus v_z$. In local form one has

$$\mathbf{\Omega} = \frac{1}{2}\Omega^{a_p}_{b_f c_f}\delta^{b_f} \bigwedge \delta^{c_f} \otimes \delta_{a_p}, (0 \leq f < p \leq z),$$

where

$$\Omega^{a_p}_{b_f c_f} = \frac{\delta N^{a_p}_{c_f}}{\partial y^{b_f}} - \frac{\partial N^{a_p}_{b_f}}{\partial y^{c_f}} + N^{<b>}_{b_f}\widetilde{N}^{a_p}_{<b>c_f} - N^{<b>}_{c_f}\widetilde{N}^{a_p}_{<b>b_f}. \tag{22}$$

The torsion $\mathbf{T}$ of d–connection $\mathbf{D}$ in $\mathcal{E}^{<z>}$ is defined by the equation

$$\mathbf{T}(\mathbf{X},\mathbf{Y}) = \mathbf{XY}^{\circ}_{\circ}\mathbf{T} \doteq D_X\mathbf{Y} - D_Y\mathbf{X} - [\mathbf{X},\mathbf{Y}]. \tag{23}$$

One holds the following h- and v$_{(p)}$−–decompositions

$$\mathbf{T}(\mathbf{X},\mathbf{Y}) = \mathbf{T}(\mathbf{hX},\mathbf{hY}) + \mathbf{T}(\mathbf{hX},\mathbf{vY}) + \mathbf{T}(\mathbf{vX},\mathbf{hY}) + \mathbf{T}(\mathbf{vX},\mathbf{vY}). \tag{24}$$

We consider the projections: $\mathbf{hT}(\mathbf{X},\mathbf{Y}), \mathbf{v}_{(p)}\mathbf{T}(\mathbf{hX},\mathbf{hY}), \mathbf{hT}(\mathbf{hX},\mathbf{hY}), \ldots$ and say that, for instance, $\mathbf{hT}(\mathbf{hX},\mathbf{hY})$ is the h(hh)-torsion of $\mathbf{D}$,
$\mathbf{v}_{(p)}\mathbf{T}(\mathbf{hX},\mathbf{hY})$ is the v$_p$(hh)-torsion of $\mathbf{D}$ and so on.

The torsion (23) is locally determined by five d-tensor fields, torsions, defined as

$$T^i_{jk} = \mathbf{hT}(\delta_k,\delta_j)\cdot d^i, \quad T^{a_p}_{jk} = \mathbf{v}_{(p)}\mathbf{T}(\delta_k,\delta_j)\cdot\delta^{a_p},$$
$$P^i_{jb_p} = \mathbf{hT}\left(\delta_{b_p},\delta_j\right)\cdot d^i, \quad P^{a_p}_{jb_f} = \mathbf{v}_{(p)}\mathbf{T}\left(\delta_{b_f},\delta_j\right)\cdot\delta^{a_p},$$
$$S^{a_p}_{b_f c_f} = \mathbf{v}_{(p)}\mathbf{T}\left(\delta_{c_f},\delta_{b_f}\right)\cdot\delta^{a_p}.$$

Using formulas (5),(6),(22) and (23) we can computer in explicit form the components of torsions (24) for a d–connection of type (16) and (17):

$$T^i_{.jk} = T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{j<a>} = C^i_{.j<a>}, T^i_{<a>j} = -C^i_{j<a>}, \tag{25}$$
$$T^i_{.j<a>} = 0, T^{<a>}_{.<b><c>} = S^{<a>}_{.<b><c>} = C^{<a>}_{<b><c>} - C^{<a>}_{<c><b>},$$
$$T^{a_p}_{.b_f c_f} = \frac{\delta N^{a_p}_{c_f}}{\partial y^{b_f}} - \frac{\delta N^{a_p}_{b_f}}{\partial y^{c_f}}, T^{<a>}_{.<b>i} = P^{<a>}_{.<b>i} = \frac{\delta N^{<a>}_i}{\partial y^{<b>}} - L^{<a>}_{.<b>j}, T^{<a>}_{.i<b>} = -P^{<a>}_{.<b>i}.$$

The curvature $\mathbf{R}$ of d–connection in $\mathcal{E}^{<z>}$ is defined by the equation

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \mathbf{XY}^{\bullet}_{\bullet}\mathbf{R}\bullet\mathbf{Z} = D_X D_Y\mathbf{Z} - D_Y D_X\mathbf{Z} - D_{[X,Y]}\mathbf{Z}. \tag{26}$$

One holds the next properties for the h- and v-decompositions of curvature:

$$\mathbf{v}_{(p)}\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{hZ} = \mathbf{0}, \ \mathbf{hR}(\mathbf{X},\mathbf{Y})\mathbf{v}_{(p)}\mathbf{Z} = \mathbf{0}, \ \mathbf{v}_{(f)}\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{v}_{(p)}\mathbf{Z} = \mathbf{0},$$
$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \mathbf{hR}(\mathbf{X},\mathbf{Y})\mathbf{hZ} + \mathbf{vR}(\mathbf{X},\mathbf{Y})\mathbf{vZ},$$

where $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_z$. From (26) and the equation $\mathbf{R}(\mathbf{X},\mathbf{Y}) = -\mathbf{R}(\mathbf{Y},\mathbf{X})$ we get that the curvature of a d-connection $\mathbf{D}$ in $\mathcal{E}^{<z>}$ is completely determined by the following d-tensor fields:

$$R^i_{h.jk} = \delta^i\cdot\mathbf{R}(\delta_k,\delta_j)\delta_h, \ R^{.<a>}_{<b>.jk} = \delta^{<a>}\cdot\mathbf{R}(\delta_k,\delta_j)\delta_{<b>}, \tag{27}$$

$$P_{j.k<c>}^{.i} = d^i \cdot \mathbf{R}\left(\delta_{<c>}, \delta_{<k>}\right)\delta_j,\ P_{<b>.<k><c>}^{.<a>} = \delta^{<a>} \cdot \mathbf{R}\left(\delta_{<c>}, \delta_{<k>}\right)\delta_{<b>},$$

$$S_{j.<b><c>}^{.i} = d^i \cdot \mathbf{R}\left(\delta_{<c>}, \delta_{<b>}\right)\delta_j,\ S_{<b>.<c><d>}^{.<a>} = \delta^{<a>} \cdot \mathbf{R}\left(\delta_{<d>}, \delta_{<c>}\right)\delta_{<b>}.$$

By a direct computation, using (5),(6),(16),(17) and (27) we get :

$$R_{h.jk}^{.i} = \frac{\delta L_{.hj}^i}{\delta x^h} - \frac{\delta L_{.hk}^i}{\delta x^j} + L_{.hj}^m L_{mk}^i - L_{.hk}^m L_{mj}^i + C_{.h<a>}^i R_{.jk}^{<a>}, \tag{28}$$

$$R_{<b>.jk}^{.<a>} = \frac{\delta L_{.<b>j}^{<a>}}{\delta x^k} - \frac{\delta L_{.<b>k}^{<a>}}{\delta x^j} + L_{.<b>j}^{<c>} L_{.<c>k}^{<a>} - L_{.<b>k}^{<c>} L_{.<c>j}^{<a>} + C_{.<b><c>}^{<a>} R_{.jk}^{<c>},$$

$$P_{j.k<a>}^{.i} = \frac{\delta L_{.jk}^i}{\partial y^{<a>}} + C_{.j<b>}^i P_{.k<a>}^{<b>} -$$

$$\left(\frac{\partial C_{.j<a>}^i}{\partial x^k} + L_{.lk}^i C_{.j<a>}^l - L_{.jk}^l C_{.l<a>}^i - L_{.<a>k}^{<c>} C_{.j<c>}^i\right),$$

$$P_{<b>.k<a>}^{.<c>} = \frac{\delta L_{.<b>k}^{<c>}}{\partial y^{<a>}} + C_{.<b><d>}^{<c>} P_{.k<a>}^{<d>} -$$

$$\left(\frac{\partial C_{.<b><a>}^{<c>}}{\partial x^k} + L_{.<d>k}^{<c>} C_{.<b><a>}^{<d>} - L_{.<b>k}^{<d>} C_{.<d><a>}^{<c>} - L_{.<a>k}^{<d>} C_{.<b><d>}^{<c>}\right),$$

$$S_{j.<b><c>}^{.i} = \frac{\delta C_{.j<b>}^i}{\partial y^{<c>}} - \frac{\delta C_{.j<c>}^i}{\partial y^{<b>}} + C_{.j<b>}^h C_{.h<c>}^i - C_{.j<c>}^h C_{h<b>}^i,$$

$$S_{<b>.<c><d>}^{.<a>} = \frac{\delta C_{.<b><c>}^{<a>}}{\partial y^{<d>}} - \frac{\delta C_{.<b><d>}^{<a>}}{\partial y^{<c>}} +$$

$$C_{.<b><c>}^{<e>} C_{.<e><d>}^{<a>} - C_{.<b><d>}^{<e>} C_{.<e><c>}^{<a>}.$$

We note that torsions (25) and curvatures (28) can be computed by particular cases of d-connections when d-connections (17), (20) or (22) are used instead of (16) and (17).

The components of the Ricci d–tensor

$$R_{<\alpha><\beta>} = R_{<\alpha>.<\beta><\tau>}^{.<\tau>}$$

with respect to locally adapted frame (6) are as follows:

$$R_{ij} = R_{i.jk}^{.k}, \quad R_{i<a>} = -{}^2P_{i<a>} = -P_{i.k<a>}^{.k}, \tag{29}$$

$$R_{<a>i} = {}^1 P_{<a>i} = P_{<a>.i<b>}^{.<b>}, \quad R_{<a><b>} = S_{<a>.<b><c>}^{.<c>}.$$

We point out that because, in general, ${}^1P_{<a>i} \neq {}^2P_{i<a>}$ the Ricci d–tensor is non symmetric.

Having defined a d–metric of type (12) in $\mathcal{E}^{<z>}$ we can introduce the scalar curvature of d–connection $\mathbf{D}$:

$$\overleftarrow{R} = G^{<\alpha><\beta>} R_{<\alpha><\beta>} = R + S,$$

where $R = g^{ij} R_{ij}$ and $S = h^{<a><b>} S_{<a><b>}$.

For our further considerations it will be also useful to use an alternative way of definition torsion (23) and curvature (26) by using the commutator

$$\Delta_{<\alpha><\beta>} \doteq \nabla_{<\alpha>}\nabla_{<\beta>} - \nabla_{<\beta>}\nabla_{<\alpha>} = 2\nabla_{[<\alpha>}\nabla_{<\beta>]}.$$

For components (25) of d–torsion we have

$$\Delta_{<\alpha><\beta>} f = T^{<\gamma>}_{.<\alpha><\beta>}\nabla_{<\gamma>} f \tag{30}$$

for every scalar function f on $\mathcal{E}^{<z>}$. Curvature can be introduced as an operator acting on arbitrary d–vector $V^{<\delta>}$:

$$(\Delta_{<\alpha><\beta>} - T^{<\gamma>}_{.<\alpha><\beta>}\nabla_{<\gamma>})V^{<\delta>} = R^{\cdot<\delta>}_{<\gamma>.<\alpha><\beta>}V^{<\gamma>} \tag{31}$$

(in this section we follow conventions of Miron and Anastasiei [22, 23] on d–tensors; we can obtain corresponding Penrose and Rindler abstract index formulas [27, 28] just for a trivial N-connection structure and by changing denotations for components of torsion and curvature in this manner: $T^{\gamma}_{.\alpha\beta} \to T_{\alpha\beta}{}^{\gamma}$ and $R^{\cdot\delta}_{\gamma.\alpha\beta} \to R_{\alpha\beta\gamma}{}^{\delta}$).

Here we also note that torsion and curvature of a d–connection on $\mathcal{E}^{<z>}$ satisfy generalized for ha–spaces Ricci and Bianchi identities which in terms of components (30) and (31) are written respectively as

$$R^{\cdot<\delta>}_{[<\gamma>.<\alpha><\beta>]} + \nabla_{[<\alpha>}T^{<\delta>}_{.<\beta><\gamma>]} + T^{<\nu>}_{.[<\alpha><\beta>}T^{<\delta>}_{.<\gamma>]<\nu>} = 0 \tag{32}$$

and

$$\nabla_{[<\alpha>}R^{\cdot<\sigma>}_{|<\nu>|<\beta><\gamma>]} + T^{<\delta>}_{\cdot[<\alpha><\beta>}R^{\cdot<\sigma>}_{|<\nu>|.<\gamma>]<\delta>} = 0.$$

Identities (32) can be proved similarly as in [27] by taking into account that indices play a distinguished character.

We can also consider a ha-generalization of the so-called conformal Weyl tensor (see, for instance, [27]) which can be written as a d-tensor in this form:

$$C^{<\gamma><\delta>}_{\quad<\alpha><\beta>} = R^{<\gamma><\delta>}_{\quad<\alpha><\beta>} - \frac{4}{n+m_1+\cdots+m_z-2}R^{[<\gamma>}_{\quad[<\alpha>}\ \delta^{<\delta>]}_{\quad<\beta>]} + \tag{33}$$

$$\frac{2}{(n+m_1+\cdots m_z-1)(n+m_1+\cdots+m_z-2)}\overleftarrow{R}\ \delta^{[<\gamma>}_{\quad[<\alpha>}\ \delta^{<\delta>]}_{\quad<\beta>]}.$$

This object is conformally invariant on ha–spaces provided with d–connection generated by d–metric structures.

2.3. Field equations for ha–gravity

The Einstein equations in some models of higher order anisotropic supergravity have been considered in [41]. Here we note that the Einstein equations and conservation laws on v–bundles provided with N-connection structures were studied in detail in [22, 23, 2, 3, 45, 44, 39]. In Ref. [42] we proved that the la-gravity can be formulated in a gauge like manner and analyzed the conditions when the Einstein la-gravitational field equations are equivalent to a corresponding form of Yang-Mills equations. Our aim here is to write the higher order anisotropic gravitational field equations in a form more convenient for theirs equivalent reformulation in ha–spinor variables.

We define d-tensor $\Phi_{<\alpha><\beta>}$ as to satisfy conditions

$$-2\Phi_{<\alpha><\beta>} \doteq R_{<\alpha><\beta>} - \frac{1}{n+m_1+\cdots+m_z}\overleftarrow{R}g_{<\alpha><\beta>}$$

which is the torsionless part of the Ricci tensor for locally isotropic spaces [27, 28], i.e. $\Phi_{<\alpha>}^{<\alpha>} \doteq 0$. The Einstein equations on ha–spaces

$$\overleftarrow{G}_{<\alpha><\beta>} + \lambda g_{<\alpha><\beta>} = \kappa E_{<\alpha><\beta>}, \qquad (34)$$

where

$$\overleftarrow{G}_{<\alpha><\beta>} = R_{<\alpha><\beta>} - \frac{1}{2}\overleftarrow{R}g_{<\alpha><\beta>}$$

is the Einstein d–tensor, λ and κ are correspondingly the cosmological and gravitational constants and by $E_{<\alpha><\beta>}$ is denoted the locally anisotropic energy–momentum d–tensor, can be rewritten in equivalent form:

$$\Phi_{<\alpha><\beta>} = -\frac{\kappa}{2}(E_{<\alpha><\beta>} - \frac{1}{n+m_1+\cdots+m_z}E_{<\tau>}^{<\tau>}\ g_{<\alpha><\beta>}). \qquad (35)$$

Because ha–spaces generally have nonzero torsions we shall add to (35) (equivalently to (34)) a system of algebraic d–field equations with the source $S_{<\beta><\gamma>}^{<\alpha>}$ being the locally anisotropic spin density of matter (if we consider a variant of higher order anisotropic Einstein–Cartan theory):

$$T_{\ <\alpha><\beta>}^{<\gamma>} + 2\delta_{\ [<\alpha>}^{<\gamma>}T_{\ <\beta>]<\delta>}^{<\delta>} = \kappa S_{\ <\alpha><\beta>.}^{<\gamma>} \qquad (36)$$

From (32) and (36) one follows the conservation law of higher order anisotropic spin matter:

$$\nabla_{<\gamma>}S_{\ <\alpha><\beta>}^{<\gamma>} - T_{\ <\delta><\gamma>}^{<\delta>}S_{\ <\alpha><\beta>}^{<\gamma>} = E_{<\beta><\alpha>} - E_{<\alpha><\beta>}.$$

Finally, we remark that all presented geometric constructions contain those elaborated for generalized Lagrange spaces [22, 23] (for which a tangent bundle TM is considered instead of a v-bundle $\mathcal{E}^{<z>}$) and for constructions on the so called osculator bundles with different prolongations and extensions of Finsler and Lagrange metrics [24]. We also note that the Lagrange (Finsler) geometry is characterized by a metric of type (12) with components parametrized as $g_{ij} = \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial y^i\partial y^j}$ $\left(g_{ij} = \frac{1}{2}\frac{\partial^2\Lambda^2}{\partial y^i\partial y^j}\right)$ and $h_{ij} = g_{ij}$, where $\mathcal{L} = \mathcal{L}\ (x,y)$ $(\Lambda = \Lambda\,(x,y))$ is a Lagrangian (Finsler metric) on TM (see details in [22, 23, 21, 9]).

3. Distinguished Clifford algebras

The typical fiber of dv-bundle ξ_d , $\pi_d : HE \oplus V_1E \oplus \cdots \oplus V_zE \to E$ is a d-vector space, $\mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus \cdots \oplus v_z\mathcal{F}$, split into horizontal $h\mathcal{F}$ and verticals $v_p\mathcal{F}, p = 1, \ldots, z$ subspaces, with metric $G(g, h)$ induced by v-bundle metric (12). Clifford algebras (see, for example, Refs. [19, 31, 28]) formulated for d-vector spaces will be called Clifford d-algebras [34, 35, 43]. We shall consider the main properties of Clifford d–algebras. The proof of theorems will be based on the technique developed in Ref. [19] correspondingly adapted to the distinguished character of spaces in consideration.

Let k be a number field (for our purposes $k = \mathcal{R}$ or $k = \mathcal{C}, \mathcal{R}$ and $\mathcal{C}$, are, respectively real and complex number fields) and define $\mathcal{F}$, as a d-vector space on k provided with nondegenerate symmetrical quadratic form (metric) G. Let C be an algebra on k (not necessarily commutative) and $j : \mathcal{F} \to C$ a homomorphism of underlying vector spaces such that $j(u)^2 = G(u) \cdot 1$ (1 is the unity in algebra C and d-vector $u \in \mathcal{F}$).

We are interested in definition of the pair (C, j) satisfying the next universitality conditions. For every k-algebra A and arbitrary homomorphism $\varphi : \mathcal{F} \to A$ of the underlying d-vector spaces, such that $(\varphi(u))^2 \to G(u) \cdot 1$, there is a unique homomorphism of algebras $\psi : C \to A$ transforming diagram 1 into a commutative one.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, leftrightarrow, "j"] \arrow[dr, "\varphi"'] & & C \arrow[dl, "\tau"] \\ & A & \end{tikzcd}$$

Diagram 1

The algebra solving this problem will be denoted as $C(\mathcal{F}, A)$ [equivalently as $C(G)$ or $C(\mathcal{F})$] and called as Clifford d–algebra associated with pair $(\mathcal{F}, G)$.

Theorem 1 *The above-presented diagram has a unique solution (C, j) up to isomorphism.*

Proof: (We adapt for d-algebras that of Ref. [19], p. 127.) For a universal problem the uniqueness is obvious if we prove the existence of solution $C(G)$. To do this we use tensor algebra $\mathcal{L}^{(F)} = \oplus\mathcal{L}^{pr}_{qs}(\mathcal{F}) = \oplus^{\infty}_{i=0}T^i(\mathcal{F})$, where $T^0(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$ for $i > 0$. Let $I(G)$ be the bilateral ideal generated by elements of form $\epsilon(u) = u \otimes u - G(u) \cdot 1$ where $u \in \mathcal{F}$ and 1 is the unity element of algebra $\mathcal{L}(\mathcal{F})$. Every element from $I(G)$ can be written as $\sum_i \lambda_i \epsilon(u_i)\mu_i$, where $\lambda_i, \mu_i \in \mathcal{L}(\mathcal{F})$ and $u_i \in \mathcal{F}$. Let $C(G) = \mathcal{L}(\mathcal{F})/I(G)$ and define $j : \mathcal{F} \to C(G)$ as the composition of monomorphism $i : \mathcal{F} \to L^1(\mathcal{F}) \subset \mathcal{L}(\mathcal{F})$ and projection $p : \mathcal{L}(\mathcal{F}) \to C(G)$. In this case pair $(C(G), j)$ is the solution of our problem.

From the general properties of tensor algebras the homomorphism $\varphi : \mathcal{F} \to A$ can be extended to $\mathcal{L}(\mathcal{F})$, i.e., the diagram 2 is commutative, where ρ is a monomorphism of algebras. Because $(\varphi(u))^2 = G(u) \cdot 1$, then ρ vanishes on ideal $I(G)$ and in this case the necessary homomorphism τ is defined. As a consequence of uniqueness of ρ, the homomorphism τ is unique.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & \mathcal{L}(\mathcal{F}) \arrow[dl, "\rho"] \\ & A & \end{tikzcd}$$

Diagram 2

Tensor d-algebra $\mathcal{L}(\mathcal{F})$ can be considered as a $\mathcal{Z}/2$ graded algebra. Really, let us introduce $\mathcal{L}^{(0)}(\mathcal{F}) = \sum_{i=1}^{\infty} T^{2i}(\mathcal{F})$ and $\mathcal{L}^{(1)}(\mathcal{F}) = \sum_{i=1}^{\infty} T^{2i+1}(\mathcal{F})$. Setting $I^{(\alpha)}(G) = I(G) \cap \mathcal{L}^{(\alpha)}(\mathcal{F})$. Define $C^{(\alpha)}(G)$ as $p\left(\mathcal{L}^{(\alpha)}(\mathcal{F})\right)$, where $p : \mathcal{L}(\mathcal{F}) \to C(G)$ is the canonical projection. Then $C(G) = C^{(0)}(G) \oplus C^{(1)}(G)$ and in consequence we obtain that the Clifford d-algebra is $\mathcal{Z}/2$ graded.

It is obvious that Clifford d-algebra functorially depends on pair $(\mathcal{F}, G)$. If $f : \mathcal{F} \to \mathcal{F}'$ is a homomorphism of k-vector spaces, such that $G'(f(u)) = G(u)$, where G and G' are, respectively, metrics on $\mathcal{F}$ and $\mathcal{F}'$, then f induces an homomorphism of d-algebras

$$C(f) : C(G) \to C(G')$$

with identities $C(\varphi \cdot f) = C(\varphi) C(f)$ and $C(Id_{\mathcal{F}}) = Id_{C(\mathcal{F})}$.

If $\mathcal{A}^{\alpha}$ and $\mathcal{B}^{\beta}$ are $\mathcal{Z}/2$–graded d–algebras, then their graded tensorial product $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ is defined as a d-algebra for k-vector d-space $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ with the graded product induced as $(a \otimes b)(c \otimes d) = (-1)^{\alpha\beta} ac \otimes bd$, where $b \in \mathcal{B}^{\alpha}$ and $c \in \mathcal{A}^{\alpha}$ $\quad (\alpha, \beta = 0, 1)$.

Now we reformulate for d–algebras the Chevalley theorem [12]:

Theorem 2 *The Clifford d-algebra*

$$C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \cdots \oplus v_z\mathcal{F}, g + h_1 + \cdots + h_z)$$

is naturally isomorphic to $C(g) \otimes C(h_1) \otimes \cdots \otimes C(h_z)$.

Proof. Let $n : h\mathcal{F} \to C(g)$ and $n'_{(p)} : v_{(p)}\mathcal{F} \to C\left(h_{(p)}\right)$ be canonical maps and map

$$m : h\mathcal{F} \oplus v_1\mathcal{F} \cdots \oplus v_z\mathcal{F} \to C(g) \otimes C(h_1) \otimes \cdots \otimes C(h_z)$$

is defined as

$$m(x, y_{(1)}, \ldots, y_{(z)}) =$$
$$n(x) \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes n'(y_{(1)}) \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes n'(y_{(z)}),$$

$x \in h\mathcal{F}, y_{(1)} \in v_{(1)}\mathcal{F}, \ldots, y_{(z)} \in v_{(z)}\mathcal{F}$. We have

$$\left(m(x, y_{(1)}, \ldots, y_{(z)})\right)^2 = \left[(n(x))^2 + \left(n'\left(y_{(1)}\right)\right)^2 + \cdots + \left(n'\left(y_{(z)}\right)\right)^2\right] \cdot 1 =$$
$$[g(x) + h\left(y_{(1)}\right) + \cdots + h\left(y_{(z)}\right)].$$

Taking into account the universality property of Clifford d-algebras we conclude that $m_1 + \cdots + m_z$ induces the homomorphism

$$\zeta : C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \cdots \oplus v_z\mathcal{F}, g + h_1 + \cdots + h_z) \to$$
$$C(h\mathcal{F}, g)\,\widehat{\otimes} C(v_1\mathcal{F}, h_1)\,\widehat{\otimes} \cdots C(v_z\mathcal{F}, h_z).$$

We also can define a homomorphism

$$\upsilon : C(h\mathcal{F}, g)\,\widehat{\otimes} C\left(v_1\mathcal{F}, h_{(1)}\right)\widehat{\otimes} \cdots \widehat{\otimes} C\left(v_z\mathcal{F}, h_{(z)}\right) \to$$

$$C\left(h\mathcal{F}\oplus v_1\mathcal{F}\oplus\cdots\oplus v_z\mathcal{F},g+h_{(1)}+\cdots+h_{(z)}\right)$$

by using formula

$$\upsilon\left(x\otimes y_{(1)}\otimes\cdots\otimes y_{(z)}\right)=\delta\left(x\right)\delta'_{(1)}\left(y_{(1)}\right)\cdots\delta'_{(z)}\left(y_{(z)}\right),$$

where homomorphisms δ and $\delta'_{(1)},\ldots,\delta'_{(z)}$ are, respectively, induced by imbeddings of $h\mathcal{F}$ and $v_1\mathcal{F}$ into $h\mathcal{F}\oplus v_1\mathcal{F}\oplus\cdots\oplus v_z\mathcal{F}$:

$$\delta : C\left(h\mathcal{F},g\right)\to C\left(h\mathcal{F}\oplus v_1\mathcal{F}\oplus\cdots\oplus v_z\mathcal{F},g+h_{(1)}+\cdots+h_{(z)}\right),$$

$$\delta'_{(1)} : C\left(v_1\mathcal{F},h_{(1)}\right)\to C\left(h\mathcal{F}\oplus v_1\mathcal{F}\oplus\cdots\oplus v_z\mathcal{F},g+h_{(1)}+\cdots+h_{(z)}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\delta'_{(z)} : C\left(v_z\mathcal{F},h_{(z)}\right)\to C\left(h\mathcal{F}\oplus v_1\mathcal{F}\oplus\cdots\oplus v_z\mathcal{F},g+h_{(1)}+\cdots+h_{(z)}\right).$$

Superpositions of homomorphisms ζ and υ lead to identities

$$\upsilon\zeta=Id_{C(h\mathcal{F},g)\widehat{\otimes}C\left(v_1\mathcal{F},h_{(1)}\right)\widehat{\otimes}\cdots\widehat{\otimes}C\left(v_z\mathcal{F},h_{(z)}\right)}, \tag{37}$$

$$\zeta\upsilon=Id_{C(h\mathcal{F},g)\widehat{\otimes}C\left(v_1\mathcal{F},h_{(1)}\right)\widehat{\otimes}\cdots\widehat{\otimes}C\left(v_z\mathcal{F},h_{(z)}\right)}.$$

Really, d-algebra

$$C\left(h\mathcal{F}\oplus v_1\mathcal{F}\oplus\cdots\oplus v_z\mathcal{F},g+h_{(1)}+\cdots+h_{(z)}\right)$$

is generated by elements of type $m(x,y_{(1)},\ldots y_{(z)})$. Calculating

$$\upsilon\zeta\left(m\left(x,y_{(1)},\ldots y_{(z)}\right)\right)=\upsilon(n\left(x\right)\otimes 1\otimes\cdots\otimes 1+1\otimes n'_{(1)}\left(y_{(1)}\right)\otimes\cdots\otimes 1+\cdots+$$
$$1\otimes\cdots\otimes n'_{(z)}\left(y_{(z)}\right))=\delta\left(n\left(x\right)\right)\delta\left(n'_{(1)}\left(y_{(1)}\right)\right)\cdots\delta\left(n'_{(z)}\left(y_{(z)}\right)\right)=$$
$$m\left(x,0,\ldots,0\right)+m(0,y_{(1)},\ldots,0)+\cdots+m(0,0,\ldots,y_{(z)})=m\left(x,y_{(1)},\ldots,y_{(z)}\right),$$

we prove the first identity in (37).

On the other hand, d-algebra

$$C\left(h\mathcal{F},g\right)\widehat{\otimes}C\left(v_1\mathcal{F},h_{(1)}\right)\widehat{\otimes}\cdots\widehat{\otimes}C\left(v_z\mathcal{F},h_{(z)}\right)$$

is generated by elements of type

$$n\left(x\right)\otimes 1\otimes\cdots\otimes, 1\otimes n'_{(1)}\left(y_{(1)}\right)\otimes\cdots\otimes 1,\cdots 1\otimes\cdots\otimes n'_{(z)}\left(y_{(z)}\right),$$

we prove the second identity in (37).

Following from the above–mentioned properties of homomorphisms ζ and υ we can assert that the natural isomorphism is explicitly constructed.□

In consequence of theorem 2 we conclude that all operations with Clifford d-algebras can be reduced to calculations for $C\left(h\mathcal{F},g\right)$ and

$C\left(v_{(p)}\mathcal{F},h_{(p)}\right)$ which are usual Clifford algebras of dimension 2^n and, respectively, 2^{m_p} [19, 8].

Of special interest is the case when $k=\mathcal{R}$ and $\mathcal{F}$ is isomorphic to vector space $\mathcal{R}^{p+q,a+b}$ provided with quadratic form $-x_1^2-\cdots-x_p^2+x_{p+q}^2-y_1^2-\cdots-y_a^2+\cdots+y_{a+b}^2$. In this case, the Clifford algebra, denoted as $\left(C^{p,q},C^{a,b}\right)$, is generated by symbols $e_1^{(x)}$, $e_2^{(x)},\ldots,e_{p+q}^{(x)},e_1^{(y)},e_2^{(y)},\ldots,e_{a+b}^{(y)}$ satisfying properties $(e_i)^2=-1\ \ (1\le i\le p),(e_j)^2=-1\ \ (1\le j\le a),(e_k)^2=1\ (p+1\le k\le p+q)$, $(e_j)^2=1\ (n+1\le s\le a+b)$, $e_ie_j=-e_je_i,\ i\neq j$. Explicit calculations of $C^{p,q}$ and $C^{a,b}$ are possible by using isomorphisms [19, 28]

$$C^{p+n,q+n}\simeq C^{p,q}\otimes M_2(\mathcal{R})\otimes\cdots\otimes M_2(\mathcal{R})\cong C^{p,q}\otimes M_{2^n}(\mathcal{R})\cong M_{2^n}(C^{p,q}),$$

where $M_s(A)$ denotes the ring of quadratic matrices of order s with coefficients in ring A. Here we write the simplest isomorphisms $C^{1,0}\simeq\mathcal{C},C^{0,1}\simeq\mathcal{R}\oplus\mathcal{R}$, and $C^{2,0}=\mathcal{H}$, where by $\mathcal{H}$ is denoted the body of quaternions. We summarize this calculus as (as in Ref. [8])

$$C^{0,0}=\mathcal{R},C^{1,0}=\mathcal{C},C^{0,1}=\mathcal{R}\oplus\mathcal{R},C^{2,0}=\mathcal{H},C^{0,2}=M_2(\mathcal{R}),$$

$$C^{3,0}=\mathcal{H}\oplus\mathcal{H},C^{0,3}=M_2(\mathcal{R}),C^{4,0}=M_2(\mathcal{H}),C^{0,4}=M_2(\mathcal{H}),$$

$$C^{5,0}=M_4(\mathcal{C}),\ C^{0,5}=M_2(\mathcal{H})\oplus M_2(\mathcal{H}),\ C^{6,0}=M_8(\mathcal{R}),\ C^{0,6}=M_4(\mathcal{H}),$$

$$C^{7,0}=M_8(\mathcal{R})\oplus M_8(\mathcal{R}),\ C^{0,7}=M_8(\mathcal{C}),\ C^{8,0}=M_{16}(\mathcal{R}),\ C^{0,8}=M_{16}(\mathcal{R}).$$

One of the most important properties of real algebras $C^{0,p}\ \ (C^{0,a})$ and $C^{p,0}\ \ (C^{a,0})$ is eightfold periodicity of $p(a)$.

Now, we emphasize that H^{2n}-spaces admit locally a structure of Clifford algebra on complex vector spaces. Really, by using almost Hermitian structure $J_\alpha{}^\beta$ and considering complex space $\mathcal{C}^n$ with nondegenarate quadratic form $\sum_{a=1}^n|z_a|^2,\ z_a\in\mathcal{C}^2$ induced locally by metric (12) (rewritten in complex coordinates $z_a=x_a+iy_a$) we define Clifford algebra $\overleftarrow{C}^n=\underbrace{\overleftarrow{C}^1\otimes\cdots\otimes\overleftarrow{C}^1}_{n}$, where $\overleftarrow{C}^1=\mathcal{C}\otimes_R\mathcal{C}=\mathcal{C}\oplus\mathcal{C}$ or in consequence, $\overleftarrow{C}^n\simeq C^{n,0}\otimes_{\mathcal{R}}\mathcal{C}\approx C^{0,n}\otimes_{\mathcal{R}}\mathcal{C}$. Explicit calculations lead to isomorphisms $\overleftarrow{C}^2=C^{0,2}\otimes_{\mathcal{R}}\mathcal{C}\approx M_2(\mathcal{R})\otimes_{\mathcal{R}}\mathcal{C}\approx M_2\left(\overleftarrow{C}^n\right)$, $C^{2p}\approx M_{2^p}(\mathcal{C})$ and $\overleftarrow{C}^{2p+1}\approx M_{2^p}(\mathcal{C})\oplus M_{2^p}(\mathcal{C})$, which show that complex Clifford algebras, defined locally for H^{2n}-spaces, have periodicity 2 on p.

Considerations presented in the proof of theorem 2 show that map $j:\mathcal{F}\to C(\mathcal{F})$ is monomorphic, so we can identify space $\mathcal{F}$ with its image in $C(\mathcal{F},G)$, denoted as $u\to\overline{u}$, if $u\in C^{(0)}(\mathcal{F},G)\ \ \left(u\in C^{(1)}(\mathcal{F},G)\right)$; then $u=\overline{u}$ (respectively, $\overline{u}=-u$).

Definition 1 *The set of elements* $u\in C(G)^*$, *where* $C(G)^*$ *denotes the multiplicative group of invertible elements of* $C(\mathcal{F},G)$ *satisfying* $\overline{u}\mathcal{F}u^{-1}\in\mathcal{F}$, *is called the twisted Clifford d-group, denoted as* $\widetilde{\Gamma}(\mathcal{F})$.

Let $\widetilde{\rho} : \widetilde{\Gamma}(\mathcal{F}) \to GL(\mathcal{F})$ be the homomorphism given by $u \to \rho\widetilde{u}$, where $\widetilde{\rho}_u(w) = \overline{u}wu^{-1}$. We can verify that $\ker \widetilde{\rho} = \mathcal{R}^*$is a subgroup in $\widetilde{\Gamma}(\mathcal{F})$.

Canonical map $j : \mathcal{F} \to C(\mathcal{F})$ can be interpreted as the linear map $\mathcal{F} \to C(\mathcal{F})^0$ satisfying the universal property of Clifford d-algebras. This leads to a homomorphism of algebras, $C(\mathcal{F}) \to C(\mathcal{F})^t$, considered by an anti-involution of $C(\mathcal{F})$ and denoted as $u \to {}^tu$. More exactly, if $u_1 \dots u_n \in \mathcal{F}$, then $t_u = u_n \dots u_1$ and ${}^t\overline{u} = \overline{{}^tu} = (-1)^n u_n \dots u_1$.

Definition 2 *The spinor norm of arbitrary* $u \in C(\mathcal{F})$ *is defined as*
$S(u) = {}^t\overline{u} \cdot u \in C(\mathcal{F})$.

It is obvious that if $u, u', u'' \in \widetilde{\Gamma}(\mathcal{F})$, then $S(u, u') = S(u)S(u')$ and $S(uu'u'') = S(u)S(u')S(u'')$. For $u, u' \in \mathcal{F}S(u) = -G(u)$ and $S(u, u') = S(u)S(u')$ $= S(uu')$.

Let us introduce the orthogonal group $O(G) \subset GL(G)$ defined by metric G on $\mathcal{F}$ and denote sets $SO(G) = \{u \in O(G), \det|u| = 1\}$, $Pin(G) = \{u \in \widetilde{\Gamma}(\mathcal{F}), S(u) = 1\}$ and $Spin(G) = Pin(G) \cap C^0(\mathcal{F})$. For $\mathcal{F} \cong \mathcal{R}^{n+m}$ we write $Spin(n_E)$. By straightforward calculations (see similar considerations in Ref. [19]) we can verify the exactness of these sequences:

$$1 \to \mathcal{Z}/2 \to Pin(G) \to O(G) \to 1,$$

$$1 \to \mathcal{Z}/2 \to Spin(G) \to SO(G) \to 0,$$

$$1 \to \mathcal{Z}/2 \to Spin(n_E) \to SO(n_E) \to 1.$$

We conclude this subsection by emphasizing that the spinor norm was defined with respect to a quadratic form induced by a metric in dv-bundle $\mathcal{E}^{<z>}$. This approach differs from those presented in Refs. [6] and [25] where the existence of spinor spaces only with antisymmetric metrics on Finsler like spaces is postulated. The introduction of Pin structures is closely related with works [11, 30] in our case considered for spaces with local anisotropy.

4. Higher order anisotropic Clifford bundles

We shall consider two variants of generalization of spinor constructions defined for d-vector spaces to the case of distinguished vector bundle spaces enabled with the structure of N-connection. The first is to use the extension to the category of vector bundles. The second is to define the Clifford fibration associated with compatible linear d-connection and metric G on a vector bundle. We shall analyze both variants.

4.1. Clifford d–module structure in dv–bundles

Because functor $\mathcal{F} \to C(\mathcal{F})$ is smooth we can extend it to the category of vector bundles of type $\xi^{<z>} = \{\pi_d : HE^{<z>} \oplus V_1E^{<z>} \oplus \dots \oplus V_zE^{<z>} \to E^{<z>}\}$. Recall that by $\mathcal{F}$ we denote the typical fiber of such bundles. For $\xi^{<z>}$ we obtain a bundle of algebras, denoted as $C(\xi^{<z>})$, such that $C(\xi^{<z>})_u = C(\mathcal{F}_u)$. Multiplication in every fibre defines a continuous map $C(\xi^{<z>}) \times C(\xi^{<z>}) \to C(\xi^{<z>})$. If $\xi^{<z>}$ is a vector bundle on number field k, the structure of the $C(\xi^{<z>})$-module, the d-module, the d-module, on $\xi^{<z>}$ is given by the continuous map $C(\xi^{<z>}) \times_E \xi^{<z>} \to \xi^{<z>}$ with every fiber $\mathcal{F}_u$ provided with the structure of the $C(\mathcal{F}_u)$ $-$module, correlated with its k-module structure, Because $\mathcal{F} \subset C(\mathcal{F})$, we have a fiber to fiber map $\mathcal{F} \times_E \xi^{<z>} \to \xi^{<z>}$, inducing on every fiber the map $\mathcal{F}_u \times_E \xi^{<z>}_{(u)} \to \xi^{<z>}_{(u)}$ ($\mathcal{R}$-linear on the first factor and k-linear on the second one). Inversely, every such bilinear map defines on $\xi^{<z>}$ the structure of the $C(\xi^{<z>})$-module by virtue of universal properties of Clifford d-algebras. Equivalently, the above-mentioned bilinear map defines a morphism of v-bundles $m : \xi^{<z>} \to HOM(\xi^{<z>}, \xi^{<z>})$ [$HOM(\xi^{<z>}, \xi^{<z>})$ denotes the bundles of homomorphisms] when $(m(u))^2 = G(u)$ on every point.

Vector bundles $\xi^{<z>}$ provided with $C(\xi^{<z>})$-structures are objects of the category with morphisms being morphisms of dv-bundles, which induce on every point $u \in \xi^{<z>}$ morphisms of $C(\mathcal{F}_u)$ $-$modules. This is a Banach category contained in the category of finite-dimensional d-vector spaces on filed k. We shall not use category formalism in this work, but point to its advantages in further formulation of new directions of K-theory (see , for example, an introduction in Ref. [19]) concerned with la-spaces.

Let us denote by $H^s(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R}))$, where $n_E = n + m_1 + \cdots + m_z$, the s-dimensional cohomology group of the algebraic sheaf of germs of continuous maps of dv-bundle $\mathcal{E}^{<z>}$ with group $GL_{n_E}(\mathcal{R})$ the group of automorphisms of $\mathcal{R}^{n_E}$ (for the language of algebraic topology see, for example, Refs. [19] and [17]). We shall also use the group $SL_{n_E}(\mathcal{R}) = \{A \subset GL_{n_E}(\mathcal{R}), \det A = 1\}$. Here we point out that cohomologies $H^s(M, Gr)$ characterize the class of a principal bundle $\pi : P \to M$ on M with structural group Gr. Taking into account that we deal with bundles distinguished by an N-connection we introduce into consideration cohomologies $H^s(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R}))$ as distinguished classes (d-classes) of bundles $\mathcal{E}^{<z>}$ provided with a global N-connection structure.

For a real vector bundle $\xi^{<z>}$ on compact base $\mathcal{E}^{<z>}$ we can define the orientation on $\xi^{<z>}$ as an element $\alpha_d \in H^1(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R}))$ whose image on map

$$H^1(\mathcal{E}^{<z>}, SL_{n_E}(\mathcal{R})) \to H^1(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R}))$$

is the d-class of bundle $\mathcal{E}^{<z>}$.

Definition 3 *The spinor structure on $\xi^{<z>}$ is defined as an element $\beta_d \in H^1(\mathcal{E}^{<z>}, Spin(n_E))$ whose image in the composition*

$$H^1(\mathcal{E}^{<z>}, Spin(n_E)) \to H^1(\mathcal{E}^{<z>}, SO(n_E)) \to H^1(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R}))$$

is the d-class of $\mathcal{E}^{<z>}$.

The above definition of spinor structures can be reformulated in terms of principal bundles. Let $\xi^{<z>}$ be a real vector bundle of rank n+m on a compact base $\mathcal{E}^{<z>}$. If there is a principal bundle P_d with structural group $SO(n_E)$ [or $Spin(n_E)$], this bundle $\xi^{<z>}$ can be provided with orientation (or spinor) structure. The bundle P_d is associated with element $\alpha_d \in H^1\left(\mathcal{E}^{<z>}, SO(n_{<z>})\right)$ [or $\beta_d \in H^1\left(\mathcal{E}^{<z>}, Spin\left(n_E\right)\right)$.

We remark that a real bundle is oriented if and only if its first Stiefel-Whitney d–class vanishes,

$$w_1\left(\xi_d\right) \in H^1\left(\xi, \mathcal{Z}/2\right) = 0,$$

where $H^1\left(\mathcal{E}^{<z>}, \mathcal{Z}/2\right)$ is the first group of Chech cohomology with coefficients in $\mathcal{Z}/2$, Considering the second Stiefel–Whitney class $w_2\left(\xi^{<z>}\right) \in H^2\left(\mathcal{E}^{<z>}, \mathcal{Z}/2\right)$ it is well known that vector bundle $\xi^{<z>}$ admits the spinor structure if and only if $w_2\left(\xi^{<z>}\right) = 0$. Finally, we emphasize that taking into account that base space $\mathcal{E}^{<z>}$ is also a v-bundle, $p: E^{<z>} \to M$, we have to make explicit calculations in order to express cohomologies $H^s\left(\mathcal{E}^{<z>}, GL_{n+m}\right)$ and $H^s\left(\mathcal{E}^{<z>}, SO\left(n+m\right)\right)$ through cohomologies $H^s\left(M, GL_n\right), H^s\left(M, SO\left(m_1\right)\right), \ldots H^s\left(M, SO\left(m_z\right)\right),$, which depends on global topological structures of spaces M and $\mathcal{E}^{<z>}$. For general bundle and base spaces this requires a cumbersome cohomological calculus.

4.2. Clifford fibration

Another way of defining the spinor structure is to use Clifford fibrations. Consider the principal bundle with the structural group Gr being a subgroup of orthogonal group $O\left(G\right)$, where G is a quadratic nondegenerate form (see(12)) defined on the base (also being a bundle space) space $\mathcal{E}^{<z>}$. The fibration associated to principal fibration $P\left(\mathcal{E}^{<z>}, Gr\right)$ with a typical fiber having Clifford algebra $C\left(G\right)$ is, by definition, the Clifford fibration $PC\left(\mathcal{E}^{<z>}, Gr\right)$. We can always define a metric on the Clifford fibration if every fiber is isometric to $PC\left(\mathcal{E}^{<z>}, G\right)$ (this result is proved for arbitrary quadratic forms G on pseudo-Riemannian bases [31]). If, additionally, $Gr \subset SO\left(G\right)$ a global section can be defined on $PC\left(G\right)$.

Let $\mathcal{P}\left(\mathcal{E}^{<z>}, Gr\right)$ be the set of principal bundles with differentiable base $\mathcal{E}^{<z>}$ and structural group Gr. If $g: Gr \to Gr'$ is an homomorphism of Lie groups and $P\left(\mathcal{E}^{<z>}, Gr\right) \subset \mathcal{P}\left(\mathcal{E}^{<z>}, Gr\right)$ (for simplicity in this subsection we shall denote mentioned bundles and sets of bundles as P, P' and respectively, $\mathcal{P}, \mathcal{P}'$), we can always construct a principal bundle with the property that there is as homomorphism $f: P' \to P$ of principal bundles which can be projected to the identity map of $\mathcal{E}^{<z>}$ and corresponds to isomorphism $g: Gr \to Gr'$. If the inverse statement also holds, the bundle P' is called as the extension of P associated to g and f is called the extension homomorphism denoted as $\widetilde{g}$.

Now we can define distinguished spinor structures on bundle spaces (compare with definition 3).

Definition 4 *Let $P \in \mathcal{P}\left(\mathcal{E}^{<z>}, O\left(G\right)\right)$ be a principal bundle. A distinguished spinor structure of P, equivalently a ds-structure of $\mathcal{E}^{<z>}$ is an extension $\widetilde{P}$ of P associated to* *homomorphism* $h : PinG \rightarrow O(G)$ *where* $O(G)$ *is the group of orthogonal rotations, generated by metric* G, *in bundle* $\mathcal{E}^{<z>}$.

So, if $\widetilde{P}$ is a spinor structure of the space $\mathcal{E}^{<z>}$, then
$\widetilde{P} \in \mathcal{P}\left(\mathcal{E}^{<z>}, PinG\right)$.

The definition of spinor structures on varieties was given in Ref.[13]. In Refs. [13] and [14] it is proved that a necessary and sufficient condition for a space time to be orientable is to admit a global field of orthonormalized frames. We mention that spinor structures can be also defined on varieties modeled on Banach spaces [1]. As we have shown similar constructions are possible for the cases when space time has the structure of a v-bundle with an N-connection.

Definition 5 *A special distinguished spinor structure, ds-structure, of principal bundle* $P = P\left(\mathcal{E}^{<z>}, SO(G)\right)$ *is a principal bundle*
$\widetilde{P} = \widetilde{P}\left(\mathcal{E}^{<z>}, SpinG\right)$ *for which a homomorphism of principal bundles* $\widetilde{p} : \widetilde{P} \rightarrow P$, *projected on the identity map of* $\mathcal{E}^{<z>}$ *and corresponding to representation*

$$R : SpinG \rightarrow SO(G),$$

is defined.

In the case when the base space variety is oriented, there is a natural bijection between tangent spinor structures with a common base. For special ds–structures we can define, as for any spinor structure, the concepts of spin tensors, spinor connections, and spinor covariant derivations.

5. Almost complex spinor structures

Almost complex structures are an important characteristic of H^{2n}-spaces and of osculator bundles $Osc^{k=2k_1}(M)$, where $k_1 = 1, 2, \ldots$. For simplicity in this subsection we restrict our analysis to the case of H^{2n}-spaces. We can rewrite the almost Hermitian metric [22, 23], H^{2n}-metric (see considerations for metrics and conditions of type (12) and correspondingly (14)), in complex form [34]:

$$G = H_{ab}(z, \xi)\, dz^a \otimes dz^b, \tag{38}$$

where

$$z^a = x^a + iy^a,\ \overline{z^a} = x^a - iy^a,\ H_{ab}(z, \overline{z}) = g_{ab}(x, y)\mid_{y=y(z,\overline{z})}^{x=x(z,\overline{z})},$$

and define almost complex spinor structures. For given metric (38) on H^{2n}-space there is always a principal bundle P^U with unitary structural group U(n) which allows us to transform H^{2n}-space into v-bundle $\xi^U \approx P^U \times_{U(n)} \mathcal{R}^{2n}$. This statement will be proved after we introduce complex spinor structures on oriented real vector bundles [19].

Let us consider momentarily $k = \mathcal{C}$ and introduce into consideration [instead of the group $Spin(n)$] the group $Spin^c \times_{\mathcal{Z}/2} U(1)$ being the factor group of the product $Spin(n) \times U(1)$ with the respect to equivalence

$$(y, z) \sim (-y, -a), \quad y \in Spin(m).$$

This way we define the short exact sequence

$$1 \to U(1) \to Spin^c(n) \stackrel{S^c}{\to} SO(n) \to 1, \tag{39}$$

where $\rho^c(y, a) = \rho^c(y)$. If λ is oriented , real, and rank n, γ-bundle $\pi : E_\lambda \to M^n$, with base M^n, the complex spinor structure, spin structure, on λ is given by the principal bundle P with structural group $Spin^c(m)$ and isomorphism $\lambda \approx P \times_{Spin^c(n)} \mathcal{R}^n$ (see (39)). For such bundles the categorial equivalence can be defined as

$$\epsilon^c : \mathcal{E}^T_{\mathcal{C}}(M^n) \to \mathcal{E}^\lambda_{\mathcal{C}}(M^n), \tag{40}$$

where $\epsilon^c(E^c) = P\triangle_{Spin^c(n)}E^c$ is the category of trivial complex bundles on $M^n, \mathcal{E}^\lambda_{\mathcal{C}}(M^n)$ is the category of complex v-bundles on M^n with action of Clifford bundle $C(\lambda)$, $P\triangle_{Spin^c(n)}$ and E^c is the factor space of the bundle product $P \times_M E^c$ with respect to the equivalence $(p, e) \sim (p\widehat{g}^{-1}, \widehat{g}e), p \in P, e \in E^c$, where $\widehat{g} \in Spin^c(n)$ acts on E by via the imbedding $Spin(n) \subset C^{0,n}$ and the natural action $U(1) \subset \mathcal{C}$ on complex v-bundle $\xi^c, E^c = tot\xi^c$, for bundle $\pi^c : E^c \to M^n$.

$$\begin{tikzcd} U(n) \arrow[rr, "i"] \arrow[dr, "\sigma"'] & & SO(2n) \\ & Spin^c(2n) \arrow[ur, "\rho^c"'] & \end{tikzcd}$$

Diagram 3

Now we return to the bundle $\xi = \mathcal{E}^{<1>}$. A real v-bundle (not being a spinor bundle) admits a complex spinor structure if and only if there exist a homomorphism $\sigma : U(n) \to Spin^c(2n)$ making the diagram 3 commutative. The explicit construction of σ for arbitrary γ-bundle is given in Refs. [19] and [8]. For H^{2n}-spaces it is obvious that a diagram similar to (40) can be defined for the tangent bundle TM^n, which directly points to the possibility of defining the ${}^cSpin$-structure on H^{2n}-spaces.

Let λ be a complex, rank n, spinor bundle with

$$\tau : Spin^c(n) \times_{\mathcal{Z}/2} U(1) \to U(1) \tag{41}$$

the homomorphism defined by formula $\tau(\lambda, \delta) = \delta^2$. For P_s being the principal bundle with fiber $Spin^c(n)$ we introduce the complex linear bundle $L(\lambda^c) = P_S \times_{Spin^c(n)} \mathcal{C}$ defined as the factor space of $P_S \times \mathcal{C}$ on equivalence relation

$$(pt, z) \sim \left(p, l(t)^{-1} z\right),$$

where $t \in Spin^c(n)$. This linear bundle is associated to complex spinor structure on λ^c.

If λ^c and $\lambda^{c'}$ are complex spinor bundles, the Whitney sum $\lambda^c \oplus \lambda^{c'}$ is naturally provided with the structure of the complex spinor bundle. This follows from the holomorphism

$$\omega' : Spin^c(n) \times Spin^c(n') \to Spin^c(n+n'), \tag{42}$$

given by formula $[(\beta, z), (\beta', z')] \to [\omega(\beta, \beta'), zz']$, where ω is the homomorphism making the diagram 4 commutative. Here, $z, z' \in U(1)$. It is obvious that $L\left(\lambda^c \oplus \lambda^{c'}\right)$ is isomorphic to $L(\lambda^c) \otimes L\left(\lambda^{c'}\right)$.

$$\begin{tikzcd}
Spin(n) \times Spin(n') \arrow[r] \arrow[d] & Spin(n+n') \arrow[d] \\
O(n) \times O(n') \arrow[r] & O(n+n')
\end{tikzcd}$$

Diagram 4

We conclude this subsection by formulating our main result on complex spinor structures for H^{2n}-spaces:

Theorem 3 *Let λ^c be a complex spinor bundle of rank n and H^{2n}-space considered as a real vector bundle $\lambda^c \oplus \lambda^{c'}$ provided with almost complex structure $J^{\alpha}{}_{\beta}$; multiplication on i is given by $\begin{pmatrix} 0 & -\delta^i_j \\ \delta^i_j & 0 \end{pmatrix}$. Then, the diagram 5 is commutative up to isomorphisms ϵ^c and $\widetilde{\varepsilon}^c$ defined as in (40), $\mathcal{H}$ is functor $E^c \to E^c \otimes L(\lambda^c)$ and $\mathcal{E}^{0,2n}_{\mathcal{C}}(M^n)$ is defined by functor $\mathcal{E}_{\mathcal{C}}(M^n) \to \mathcal{E}^{0,2n}_{\mathcal{C}}(M^n)$ given as correspondence $E^c \to \Lambda(\mathcal{C}^n) \otimes E^c$ (which is a categorial equivalence), $\Lambda(\mathcal{C}^n)$ is the exterior algebra on $\mathcal{C}^n$. W is the real bundle $\lambda^c \oplus \lambda^{c'}$ provided with complex structure.*

$$\begin{tikzcd}
\mathcal{E}^{0,2n}_{\mathcal{C}}(M^{2n}) \arrow[rr, "\epsilon^c"'] \arrow[dr, "\widetilde{\varepsilon}^c"'] & & \mathcal{E}^{\lambda^c \oplus \lambda^c}_{\mathcal{C}}(M^n) \arrow[dl, "\mathcal{H}"] \\
& \mathcal{E}^{W}_{\mathcal{C}}(M^n) &
\end{tikzcd}$$

Diagram 5

Proof: We use composition of homomorphisms

$$\mu : Spin^c(2n) \xrightarrow{\pi} SO(n) \xrightarrow{r} U(n) \xrightarrow{\sigma} Spin^c(2n) \times_{\mathcal{Z}/2} U(1),$$

commutative diagram 6 and introduce composition of homomorphisms

$$\mu : Spin^c(n) \xrightarrow{\Delta} Spin^c(n) \times Spin^c(n) \xrightarrow{\omega^c} Spin^c(n),$$

where Δ is the diagonal homomorphism and ω^c is defined as in (42). Using homomorphisms (41) and (42) we obtain formula $\mu(t) = \mu(t)\, r(t)$.

$$
\begin{tikzcd}
Spin(2n) & \subset & Spin^c(2n) \\
SO(n) \arrow[u, "\beta"] \arrow[rr] & & SO(2n) \arrow[u]
\end{tikzcd}
$$

Diagram 6

Now consider bundle $P \times_{Spin^c(n)} Spin^c(2n)$ as the principal $Spin^c(2n)$-bundle, associated to $M \oplus M$ being the factor space of the product $P \times Spin^c(2n)$ on the equivalence relation $(p,t,h) \sim \left(p, \mu(t)^{-1} h\right)$. In this case the categorial equivalence (40) can be rewritten as

$$\epsilon^c(E^c) = P \times_{Spin^c(n)} Spin^c(2n) \Delta_{Spin^c(2n)} E^c$$

and seen as factor space of $P \times Spin^c(2n) \times_M E^c$ on equivalence relation

$$(pt, h, e) \sim \left(p, \mu(t)^{-1} h, e\right) \text{ and } (p, h_1, h_2, e) \sim \left(p, h_1, h_2^{-1} e\right)$$

(projections of elements p and e coincides on base M). Every element of $\epsilon^c(E^c)$ can be represented as $P\Delta_{Spin^c(n)}E^c$, i.e., as a factor space $P\Delta E^c$ on equivalence relation $(pt, e) \sim (p, \mu^c(t), e)$, when $t \in Spin^c(n)$. The complex line bundle $L(\lambda^c)$ can be interpreted as the factor space of
$P \times_{Spin^c(n)} \mathcal{C}$ on equivalence relation $(pt, \delta) \sim \left(p, r(t)^{-1} \delta\right)$.

Putting $(p,e) \otimes (p,\delta)\,(p,\delta e)$ we introduce morphism

$$\epsilon^c(E) \times L(\lambda^c) \to \epsilon^c(\lambda^c)$$

with properties $(pt, e) \otimes (pt, \delta) \to (pt, \delta e) = \left(p, \mu^c(t)^{-1} \delta e\right)$,

$\left(p, \mu^c(t)^{-1} e\right) \otimes \left(p, l(t)^{-1} e\right) \to \left(p, \mu^c(t)\, r(t)^{-1} \delta e\right)$ pointing to the fact that we have defined the isomorphism correctly and that it is an isomorphism on every fiber. □

6. D–spinor techniques

The purpose of this subsection is to show how a corresponding abstract spinor technique entailing notational and calculations advantages can be developed for arbitrary splits of dimensions of a d-vector space $\mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus \cdots \oplus v_z\mathcal{F}$, where $\dim h\mathcal{F} = n$ and $\dim v_p\mathcal{F} = m_p$. For convenience we shall also present some necessary coordinate expressions.

The problem of a rigorous definition of spinors on la-spaces (la-spinors, d-spinors) was posed and solved [34, 35, 43] in the framework of the formalism of Clifford and spinor structures on v-bundles provided with compatible nonlinear and distinguished connections and metric. We introduced d-spinors as corresponding objects of the Clifford d-algebra $\mathcal{C}(\mathcal{F}, G)$, defined for a d-vector space $\mathcal{F}$ in a standard manner (see, for instance, [19]) and proved that operations with $\mathcal{C}(\mathcal{F}, G)$ can be reduced to calculations for $\mathcal{C}(h\mathcal{F}, g), \mathcal{C}(v_1\mathcal{F}, h_1), \ldots$ and $\mathcal{C}(v_z\mathcal{F}, h_z)$, which are usual Clifford algebras of respective dimensions $2^n, 2^{m_1}, \ldots$ and 2^{m_z} (if it is necessary we can use quadratic forms g and h_p correspondingly induced on $h\mathcal{F}$ and $v_p\mathcal{F}$ by a metric $\mathbf{G}$ (12)). Considering the orthogonal subgroup $O(\mathbf{G}) \subset GL(\mathbf{G})$ defined by a metric $\mathbf{G}$ we can define the d-spinor norm and parametrize d-spinors by ordered pairs of elements of Clifford algebras $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v_p\mathcal{F}, h_p), p = 1, 2, \ldots z$. We emphasize that the splitting of a Clifford d-algebra associated to a dv-bundle $\mathcal{E}^{<z>}$ is a straightforward consequence of the global decomposition (3) defining a N-connection structure in $\mathcal{E}^{<z>}$.

In this subsection we shall omit detailed proofs which in most cases are mechanical but rather tedious. We can apply the methods developed in [26, 27, 28, 20] in a straightforward manner on h- and v-subbundles in order to verify the correctness of affirmations.

6.1. Clifford d–algebra, d–spinors and d–twistors

In order to relate the succeeding constructions with Clifford d-algebras [34, 35] we consider a la-frame decomposition of the metric (12):

$$G_{<\alpha><\beta>}(u) = l^{<\widehat{\alpha}>}_{<\alpha>}(u)\, l^{<\widehat{\beta}>}_{<\beta>}(u)\, G_{<\widehat{\alpha}><\widehat{\beta}>},$$

where the frame d-vectors and constant metric matrices are distinguished as

$$l^{<\widehat{\alpha}>}_{<\alpha>}(u) = \begin{pmatrix} l^{\widehat{j}}_{j}(u) & 0 & \cdots & 0 \\ 0 & l^{\widehat{a}_1}_{a_1}(u) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & l^{\widehat{a}_z}_{a_z}(u) \end{pmatrix},$$

$$G_{<\widehat{\alpha}><\widehat{\beta}>} = \begin{pmatrix} g_{\widehat{i}\widehat{j}} & 0 & \cdots & 0 \\ 0 & h_{\widehat{a}_1\widehat{b}_1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & h_{\widehat{a}_z\widehat{b}_z} \end{pmatrix},$$

$g_{\widehat{i}\widehat{j}}$ and $h_{\widehat{a}_1\widehat{b}_1}, \ldots, h_{\widehat{a}_z\widehat{b}_z}$ are diagonal matrices with $g_{\widehat{i}\widehat{i}} = h_{\widehat{a}_1\widehat{a}_1} = \cdots = h_{\widehat{a}_z\widehat{b}_z} = \pm 1$.

To generate Clifford d-algebras we start with matrix equations

$$\sigma_{<\widehat{\alpha}>}\sigma_{<\widehat{\beta}>} + \sigma_{<\widehat{\beta}>}\sigma_{<\widehat{\alpha}>} = -G_{<\widehat{\alpha}><\widehat{\beta}>} I, \tag{43}$$

where I is the identity matrix, matrices $\sigma_{<\widehat{\alpha}>}$ (σ-objects) act on a d-vector space $\mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus \cdots \oplus v_z\mathcal{F}$ and theirs components are distinguished as

$$\sigma_{<\widehat{\alpha}>} = \left\{ (\sigma_{<\widehat{\alpha}>})^{\cdot\gamma}_{\underline{\beta}} = \begin{pmatrix} (\sigma_{\widehat{i}})^{\cdot\underline{k}}_{\underline{j}} & 0 & \cdots & 0 \\ 0 & (\sigma_{\widehat{a}_1})^{\cdot\underline{c}_1}_{\underline{b}_1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (\sigma_{\widehat{a}_z})^{\cdot\underline{c}_z}_{\underline{b}_z} \end{pmatrix} \right\}, \tag{44}$$

indices $\underline{\beta},\underline{\gamma},\ldots$ refer to spin spaces of type $\mathcal{S} = S_{(h)} \oplus S_{(v_1)} \oplus \cdots \oplus S_{(v_z)}$ and underlined Latin indices $\underline{j},\underline{k},\ldots$ and $\underline{b}_1, \underline{c}_1, \ldots, \underline{b}_z, \underline{c}_z \ldots$ refer respectively to h-spin space $\mathcal{S}_{(h)}$ and v_p-spin space $\mathcal{S}_{(v_p)}, (p = 1, 2, \ldots, z)$ which are correspondingly associated to a h- and v_p-decomposition of a dv-bundle $\mathcal{E}^{<z>}$. The irreducible algebra of matrices $\sigma_{<\widehat{\alpha}>}$ of minimal dimension $N \times N$, where $N = N_{(n)} + N_{(m_1)} + \cdots + N_{(m_z)}$, $\dim \mathcal{S}_{(h)}=N_{(n)}$ and $\dim \mathcal{S}_{(v_p)}=N_{(m_p)}$, has these dimensions

$$N_{(n)} = \begin{cases} 2^{(n-1)/2}, & n = 2k+1 \\ 2^{n/2}, & n = 2k; \end{cases}$$

and

$$N_{(m_p)} = \begin{cases} 2^{(m_p-1)/2}, & m_p = 2k_p+1 \\ 2^{m_p}, & m_p = 2k_p, \end{cases}$$

where $k = 1, 2, \ldots, k_p = 1, 2, \ldots$

The Clifford d-algebra is generated by sums on $n+1$ elements of form

$$A_1 I + B^{\widehat{i}}\sigma_{\widehat{i}} + C^{\widehat{ij}}\sigma_{\widehat{ij}} + D^{\widehat{ijk}}\sigma_{\widehat{ijk}} + \cdots \tag{45}$$

and sums of $m_p + 1$ elements of form

$$A_{2(p)} I + B^{\widehat{a}_p}\sigma_{\widehat{a}_p} + C^{\widehat{a}_p\widehat{b}_p}\sigma_{\widehat{a}_p\widehat{b}_p} + D^{\widehat{a}_p\widehat{b}_p\widehat{c}_p}\sigma_{\widehat{a}_p\widehat{b}_p\widehat{c}_p} + \cdots$$

with antisymmetric coefficients $C^{\widehat{ij}} = C^{[\widehat{ij}]}, C^{\widehat{a}_p\widehat{b}_p} = C^{[\widehat{a}_p\widehat{b}_p]}, D^{\widehat{ijk}} = D^{[\widehat{ijk}]}, D^{\widehat{a}_p\widehat{b}_p\widehat{c}_p} = D^{[\widehat{a}_p\widehat{b}_p\widehat{c}_p]}, \ldots$ and matrices $\sigma_{\widehat{ij}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}]}, \sigma_{\widehat{a}_p\widehat{b}_p} = \sigma_{[\widehat{a}_p}\sigma_{\widehat{b}_p]}, \sigma_{\widehat{ijk}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}}\sigma_{\widehat{k}]}, \ldots$. Really, we have 2^{n+1} coefficients $(A_1, C^{\widehat{ij}}, D^{\widehat{ijk}}, \ldots)$ and 2^{m_p+1} coefficients $(A_{2(p)}, C^{\widehat{a}_p\widehat{b}_p}, D^{\widehat{a}_p\widehat{b}_p\widehat{c}_p}, \ldots)$ of the Clifford algebra on $\mathcal{F}$.

For simplicity, we shall present the necessary geometric constructions only for h-spin spaces $\mathcal{S}_{(h)}$ of dimension $N_{(n)}$. Considerations for a v-spin space $\mathcal{S}_{(v)}$ are similar but with proper characteristics for a dimension $N_{(m)}$.

In order to define the scalar (spinor) product on $\mathcal{S}_{(h)}$ we introduce into consideration this finite sum (because of a finite number of elements $\sigma_{[\widehat{ij}\cdots\widehat{k}]}$) :

$${}^{(\pm)}E^{\underline{ij}}_{\underline{km}} = \delta^{\underline{i}}_{\underline{k}}\delta^{\underline{j}}_{\underline{m}} + \frac{2}{1!}(\sigma_{\widehat{i}})^{\underline{i}}_{\underline{k}}(\sigma^{\widehat{i}})^{\underline{j}}_{\underline{m}} + \frac{2^2}{2!}(\sigma_{\widehat{ij}})^{\underline{i}}_{\underline{k}}(\sigma^{\widehat{ij}})^{\underline{j}}_{\underline{m}} + \frac{2^3}{3!}(\sigma_{\widehat{ijk}})^{\underline{i}}_{\underline{k}}(\sigma^{\widehat{ijk}})^{\underline{j}}_{\underline{m}} + \cdots \tag{46}$$

which can be factorized as

$${}^{(\pm)}E^{\underline{ij}}_{\underline{km}} = N_{(n)}\ {}^{(\pm)}\epsilon_{\underline{km}}\ {}^{(\pm)}\epsilon^{\underline{ij}} \text{ for } n = 2k \tag{47}$$

and

$${}^{(+)}E^{\underline{ij}}_{\underline{km}} = 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}},\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 0 \text{ for } n = 3(mod4), \tag{48}$$

$${}^{(+)}E^{\underline{ij}}_{\underline{km}} = 0,\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}} \text{ for } n = 1(mod4).$$

Antisymmetry of $\sigma_{\widetilde{ijk}\ldots}$ and the construction of the objects (45), (46), (47) and (48) define the properties of ϵ-objects ${}^{(\pm)}\epsilon_{\underline{km}}$ and $\epsilon_{\underline{km}}$ which have an eight-fold periodicity on n (see details in [28] and, with respect to la-spaces, [34]).

For even values of n it is possible the decomposition of every h-spin space $\mathcal{S}_{(h)}$into irreducible h-spin spaces $\mathbf{S}_{(h)}$ and $\mathbf{S}'_{(h)}$ (one considers splitting of h-indices, for instance, $\underline{l}= L\oplus L', \underline{m} = M\oplus M', \ldots$; for v$_p$-indices we shall write $\underline{a}_p = A_p\oplus A'_p, \underline{b}_p = B_p\oplus B'_p, \ldots$) and defines new ϵ-objects

$$\epsilon^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\epsilon^{\underline{lm}} +{}^{(-)} \epsilon^{\underline{lm}}\right) \text{ and } \tilde{\epsilon}^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\epsilon^{\underline{lm}} -{}^{(-)} \epsilon^{\underline{lm}}\right) \tag{49}$$

We shall omit similar formulas for ϵ-objects with lower indices.

We can verify, by using expressions (48) and straightforward calculations, these parametrizations on symmetry properties of ϵ-objects (49)

$$\epsilon^{\underline{lm}} = \begin{pmatrix} \epsilon^{LM} = \epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\epsilon}^{LM} = \tilde{\epsilon}^{ML} \end{pmatrix} \text{ for } n = 0(mod8); \tag{50}$$

$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and}$$

$$\tilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \tilde{\epsilon}^{\underline{ml}} \text{ for } n = 1(mod8);$$

$$\epsilon^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \tilde{\epsilon}^{LM'} = -\epsilon^{M'L} \\ 0 & 0 \end{pmatrix} \text{ for } n = 2(mod8);$$

$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}} = -\epsilon^{\underline{ml}}, \text{ where } {}^{(-)}\epsilon^{\underline{lm}} = 0, \text{ and}$$

$$\tilde{\epsilon}^{\underline{lm}} = \frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}} = -\tilde{\epsilon}^{\underline{ml}} \text{ for } n = 3(mod8);$$

$$\epsilon^{\underline{lm}} = \begin{pmatrix} \epsilon^{LM} = -\epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\epsilon}^{LM} = -\tilde{\epsilon}^{ML} \end{pmatrix}$$

for $n = 4(mod8)$;

$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = -\epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and}$$

$$\tilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = -\tilde{\epsilon}^{\underline{ml}} \text{ for } n = 5(mod8);$$

$$\epsilon^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \tilde{\epsilon}^{LM'} = \epsilon^{M'L} \\ 0 & 0 \end{pmatrix} \text{ for } n = 6(mod8);$$

$$\epsilon^{\underline{lm}} = \frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and}$$

$$\tilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \tilde{\epsilon}^{\underline{ml}} \text{ for } n = 7(mod8).$$

Let denote reduced and irreducible h-spinor spaces in a form pointing to the symmetry of spinor inner products in dependence of values $n = 8k + l$ $(k = 0, 1, 2, \ldots; l = 1, 2, \ldots 7)$ of the dimension of the horizontal subbundle (we shall write respectively $\triangle$ and $\circ$ for antisymmetric and symmetric inner products of reduced spinors and $\diamondsuit = (\triangle, \circ)$ and $\widetilde{\diamondsuit} = (\circ, \triangle)$ for corresponding parametrizations of inner products, in brief *i.p.*, of irreducible spinors; properties of scalar products of spinors are defined by ϵ-objects (50); we shall use $\diamond$ for a general *i.p.* when the symmetry is not pointed out):

$$\mathcal{S}_{(h)}\ (8k) = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ; \tag{51}$$

$$\mathcal{S}_{(h)}\ (8k+1) = \mathcal{S}^{(-)}_\circ \ (i.p. \text{ is defined by an } {}^{(-)}\epsilon\text{-object});$$

$$\mathcal{S}_{(h)}\ (8k+2) = \left\{ \begin{array}{c} \mathcal{S}_\diamond = (\mathbf{S}_\diamond, \mathbf{S}_\diamond), \text{ or} \\ \mathcal{S}'_\diamond = (\mathbf{S}'_{\widetilde{\diamond}}, \mathbf{S}'_{\widetilde{\diamond}}); \end{array} \right.$$

$$\mathcal{S}_{(h)}\,(8k+3) = \mathcal{S}^{(+)}_\triangle \ (i.p. \text{ is defined by an } {}^{(+)}\epsilon\text{-object});$$

$$\mathcal{S}_{(h)}\,(8k+4) = \mathbf{S}_\triangle \oplus \mathbf{S}'_\triangle;$$

$$\mathcal{S}_{(h)}\,(8k+5) = \mathcal{S}^{(-)}_\triangle \ (i.p. \text{ is defined by an } {}^{(-)}\epsilon\text{-object}),$$

$$\mathcal{S}_{(h)}\,(8k+6) = \left\{ \begin{array}{c} \mathcal{S}_\diamond = (\mathbf{S}_\diamond, \mathbf{S}_\diamond), \text{ or} \\ \mathcal{S}'_\diamond = (\mathbf{S}'_{\widetilde{\diamond}}, \mathbf{S}'_{\widetilde{\diamond}}); \end{array} \right.$$

$$\mathcal{S}_{(h)}\,(8k+7) = \mathcal{S}^{(+)}_\circ \ (i.p. \ \text{ is defined by an } {}^{(+)}\epsilon\text{-object}).$$

We note that by using corresponding ϵ-objects we can lower and rise indices of reduced and irreducible spinors (for $n = 2, 6(mod 4)$ we can exclude primed indices, or inversely, see details in [26, 27, 28]).

The similar v-spinor spaces are denoted by the same symbols as in (51) provided with a left lower mark "$|$" and parametrized with respect to the values $m = 8k' + l$ (k'=0,1,...; l=1,2,...,7) of the dimension of the vertical subbundle, for example, as

$$\mathcal{S}_{(v_p)}(8k') = \mathbf{S}_{|\circ} \oplus \mathbf{S}'_{|\circ}, \mathcal{S}_{(v_p)}\,(8k+1) = \mathcal{S}^{(-)}_{|\circ}, \ldots \tag{52}$$

We use "$\sim$"-overlined symbols,

$$\widetilde{\mathcal{S}}_{(h)}\,(8k) = \widetilde{\mathbf{S}}_\circ \oplus \widetilde{S}'_\circ, \widetilde{\mathcal{S}}_{(h)}\,(8k+1) = \widetilde{\mathcal{S}}^{(-)}_\circ, \ldots \tag{53}$$

and

$$\widetilde{\mathcal{S}}_{(v_p)}(8k') = \widetilde{\mathbf{S}}_{|\circ} \oplus \widetilde{S}'_{|\circ}, \widetilde{\mathcal{S}}_{(v_p)}\,(8k'+1) = \widetilde{\mathcal{S}}^{(-)}_{|\circ}, \ldots \tag{54}$$

respectively for the dual to (50) and (51) spinor spaces.

The spinor spaces (50),(52), (53) and (54) are called the prime spinor spaces, in brief p-spinors. They are considered as building blocks of distinguished, for simplicity we consider (n, m_1)–spinor spaces constructed in this manner:

$$\mathcal{S}(_{\circ\circ,\circ\circ}) = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \mathbf{S}_{|\circ} \oplus \mathbf{S}'_{|\circ}, \mathcal{S}(_{\circ\circ,\circ}|^\circ) = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \mathbf{S}_{|\circ} \oplus \widetilde{\mathbf{S}}'_{|\circ}, \tag{55}$$

$$\mathcal{S}(_{\circ\circ,}|^{\circ\circ}) = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \widetilde{\mathbf{S}}_{|\circ} \oplus \widetilde{\mathbf{S}}'_{|\circ}, \mathcal{S}(_{\circ}|^{\circ\circ\circ}) = \mathbf{S}_\circ \oplus \widetilde{\mathbf{S}}'_\circ \oplus \widetilde{\mathbf{S}}_{|\circ} \oplus \widetilde{\mathbf{S}}'_{|\circ},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\mathcal{S}(_{\triangle,\triangle}) = \mathcal{S}^{(+)}_{\triangle} \oplus S^{(+)}_{|\triangle}, S(_{\triangle,}{}^{\triangle}) = \mathcal{S}^{(+)}_{\triangle} \oplus \widetilde{S}^{(+)}_{|\triangle},$$

$$\dots\dots\dots\dots\dots$$

$$\mathcal{S}(_{\triangle}|^{\circ},_{\diamondsuit}) = \mathbf{S}_\triangle \oplus \widetilde{S_\circ}' \oplus \mathcal{S}_{|\diamondsuit}, \mathcal{S}(_{\triangle}|^{\circ},{}^{\diamondsuit}) = \mathbf{S}_\triangle \oplus \widetilde{S_\circ}' \oplus \widetilde{\mathcal{S}}^{\diamondsuit}_{|},$$

$$\dots\dots\dots\dots\dots$$

Considering the operation of dualization of prime components in (55) we can generate different isomorphic variants of distinguished (n, m_1)-spinor spaces. If we add anisotropic "shalls" with $m_2, \dots, m_z$, we have to extend correspondingly spaces (55), for instance,

$$\mathcal{S}(_{\circ\circ,\circ\circ(1)}, \dots,_{\infty(p)}, \dots,_{\infty(z)}) = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \mathbf{S}_{|(\mathbf{1})\circ} \oplus \mathbf{S}'_{|(\mathbf{1})\circ} \oplus \cdots$$

$$\oplus \mathbf{S}_{|(\mathbf{p})\circ} \oplus \mathbf{S}'_{|(\mathbf{p})\circ} \oplus \dots \oplus \mathbf{S}_{|(\mathbf{z})\circ} \oplus \mathbf{S}'_{|(\mathbf{z})\circ},$$

and so on.

We define a d-spinor space $\mathcal{S}_{(n,m_1)}$ as a direct sum of a horizontal and a vertical spinor spaces of type (55), for instance,

$$\mathcal{S}_{(8k,8k')} = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \mathbf{S}_{|\circ} \oplus \mathbf{S}'_{|\circ}, \mathcal{S}_{(8k,8k'+1)} = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ \oplus \mathcal{S}^{(-)}_{|\circ}, \dots,$$

$$\mathcal{S}_{(8k+4,8k'+5)} = \mathbf{S}_\triangle \oplus \mathbf{S}'_\triangle \oplus \mathcal{S}^{(-)}_{|\triangle}, \dots$$

The scalar product on a $\mathcal{S}_{(n,m_1)}$ is induced by (corresponding to fixed values of n and m_1) ϵ-objects (50) considered for h- and v_1-components. We present also an example for $\mathcal{S}_{(n,m_1+\dots+m_z)}$:

$$\mathcal{S}_{(8k+4,8k_{(1)}+5,\dots,8k_{(p)}+4,\dots 8k_{(z)})} =$$

$$\mathbf{S}_\triangle \oplus \mathbf{S}'_\triangle \oplus \mathcal{S}^{(-)}_{|(1)\triangle} \oplus \dots \oplus \mathbf{S}_{|(p)\triangle} \oplus \mathbf{S}'_{|(p)\triangle} \oplus \dots \oplus \mathbf{S}_{|(z)\circ} \oplus \mathbf{S}'_{|(z)\circ}.$$

Having introduced d-spinors for dimensions $(n, m_1 + \dots + m_z)$ we can write out the generalization for ha–spaces of twistor equations [27] by using the distinguished σ-objects (44):

$$(\sigma_{(<\widehat{\alpha}>)})^{\cdot\cdot\gamma}_{|\underline{\beta}|} \quad \frac{\delta\omega^{\underline{\beta}}}{\delta u^{<\widehat{\beta}>)}} = \frac{1}{n + m_1 + \dots + m_z} \quad G_{<\widehat{\alpha}><\widehat{\beta}>}(\sigma^{\widehat{\epsilon}})^{\cdot\cdot\gamma}_{\underline{\beta}} \quad \frac{\delta\omega^{\underline{\beta}}}{\delta u^{\widehat{\epsilon}}}, \tag{56}$$

where $\left|\underline{\beta}\right|$ denotes that we do not consider symmetrization on this index. The general solution of (56) on the d-vector space $\mathcal{F}$ looks like as

$$\omega^{\underline{\beta}} = \Omega^{\underline{\beta}} + u^{<\widehat{\alpha}>}(\sigma_{<\widehat{\alpha}>})^{\cdot\cdot\underline{\beta}}_{\underline{\epsilon}}\Pi^{\underline{\epsilon}}, \tag{57}$$

where $\Omega^{\underline{\beta}}$ and $\Pi^{\underline{\epsilon}}$ are constant d-spinors. For fixed values of dimensions n and $m = m_1 + \cdots m_z$ we mast analyze the reduced and irreducible components of h- and v_p-parts of equations (56) and their solutions (57) in order to find the symmetry properties of a d-twistor $\mathbf{Z}^{\alpha}$ defined as a pair of d-spinors

$$\mathbf{Z}^{\alpha} = (\omega^{\underline{\alpha}}, \pi'_{\underline{\beta}}),$$

where $\pi_{\underline{\beta}'} = \pi^{(0)}_{\underline{\beta}'} \in \widetilde{\mathcal{S}}_{(n,m_1,\ldots,m_z)}$ is a constant dual d-spinor. The problem of definition of spinors and twistors on ha-spaces was firstly considered in [43] (see also [32, 33]) in connection with the possibility to extend the equations (56) and theirs solutions (57), by using nearly autoparallel maps, on curved, locally isotropic or anisotropic, spaces. We note that the definition of twistors have been extended to higher order anisotropic spaces with trivial N– and d–connections.

6.2. Mutual transforms of d-tensors and d-spinors

The spinor algebra for spaces of higher dimensions can not be considered as a real alternative to the tensor algebra as for locally isotropic spaces of dimensions $n = 3, 4$ [26, 27, 28]. The same holds true for ha-spaces and we emphasize that it is not quite convenient to perform a spinor calculus for dimensions $n, m >> 4$. Nevertheless, the concept of spinors is important for every type of spaces, we can deeply understand the fundamental properties of geometrical objects on ha-spaces, and we shall consider in this subsection some questions concerning transforms of d-tensor objects into d-spinor ones.

6.3. Transformation of d-tensors into d-spinors

In order to pass from d-tensors to d-spinors we must use σ-objects (44) written in reduced or irreduced form (in dependence of fixed values of dimensions n and m):

$$(\sigma_{<\widehat{\alpha}>})^{\cdot\gamma}_{\underline{\beta}},\ (\sigma^{<\widehat{\alpha}>})^{\underline{\beta\gamma}},\ (\sigma^{<\widehat{\alpha}>})_{\underline{\beta\gamma}},\ldots,$$
$$(\sigma_{<\widehat{a}>})^{\underline{bc}},\ldots,(\sigma_{\widehat{i}})_{\underline{jk}},\ldots,(\sigma_{<\widehat{a}>})^{AA'},\ldots,(\sigma^{\widehat{i}})_{II'},\ldots \quad (58)$$

It is obvious that contracting with corresponding σ-objects (58) we can introduce instead of d-tensors indices the d-spinor ones, for instance,

$$\omega^{\underline{\beta\gamma}} = (\sigma^{<\widehat{\alpha}>})^{\underline{\beta\gamma}}\omega_{<\widehat{\alpha}>}, \quad \omega_{AB'} = (\sigma^{<\widehat{a}>})_{AB'}\omega_{<\widehat{a}>}, \quad \ldots, \zeta^{\underline{i}}_{\cdot\underline{j}} = (\sigma^{\widehat{k}})^{\underline{i}}_{\cdot\underline{j}}\zeta_{\widehat{k}}, \ldots$$

For d-tensors containing groups of antisymmetric indices there is a more simple procedure of theirs transforming into d-spinors because the objects

$$(\sigma_{\widehat{\alpha}\widehat{\beta}\cdots\widehat{\gamma}})^{\underline{\delta\nu}}, \quad (\sigma^{\widehat{a}\widehat{b}\cdots\widehat{c}})^{\underline{de}}, \quad \ldots, (\sigma^{\widehat{i}\widehat{j}\cdots\widehat{k}})_{II'}, \quad \ldots \quad (59)$$

can be used for sets of such indices into pairs of d-spinor indices. Let us enumerate some properties of σ-objects of type (59) (for simplicity we consider only h-components having q indices $\widehat{i}, \widehat{j}, \widehat{k}, \ldots$ taking values from 1 to n; the properties of v_p-components can be written in a similar manner with respect to indices $\widehat{a}_p, \widehat{b}_p, \widehat{c}_p \ldots$ taking values from 1 to m):

$$(\sigma_{\widehat{i}\cdots\widehat{j}})^{\underline{kl}} \text{ is } \left\{ \begin{array}{c} \text{symmetric on } \underline{k}, \underline{l} \text{ for } n-2q \equiv 1,7 \ (mod\ 8); \\ \text{antisymmetric on } \underline{k}, \underline{l} \text{ for } n-2q \equiv 3,5 \ (mod\ 8) \end{array} \right\} \tag{60}$$

for odd values of n, and an object

$$(\sigma_{\widehat{i}\cdots\widehat{j}})^{IJ} \ \left((\sigma_{\widehat{i}\cdots\widehat{j}})^{I'J'}\right)$$

$$\text{is } \left\{ \begin{array}{c} \text{symmetric on } I,J\ (I',J') \text{ for } n-2q \equiv 0 \ (mod\ 8); \\ \text{antisymmetric on } I,J\ (I',J') \text{ for } n-2q \equiv 4 \ (mod\ 8) \end{array} \right\} \tag{61}$$

or

$$(\sigma_{\widehat{i}\cdots\widehat{j}})^{IJ'} = \pm(\sigma_{\widehat{i}\cdots\widehat{j}})^{J'I} \left\{ \begin{array}{c} n+2q \equiv 6(mod 8); \\ n+2q \equiv 2(mod 8), \end{array} \right. \tag{62}$$

with vanishing of the rest of reduced components of the d-tensor $(\sigma_{\widehat{i}\cdots\widehat{j}})^{\underline{kl}}$ with prime/unprime sets of indices.

6.4. Transformation of d-spinors into d-tensors; fundamental d-spinors

We can transform every d-spinor $\xi^{\underline{\alpha}} = (\xi^{\underline{i}}, \xi^{\underline{a}_1}, \ldots, \xi^{\underline{a}_z})$ into a corresponding d-tensor. For simplicity, we consider this construction only for a h-component $\xi^{\underline{i}}$ on a h-space being of dimension n. The values

$$\xi^{\underline{\alpha}}\xi^{\underline{\beta}}(\sigma^{\widehat{i}\cdots\widehat{j}})_{\underline{\alpha\beta}} \quad (n \text{ is odd}) \tag{63}$$

or

$$\xi^{I}\xi^{J}(\sigma^{\widehat{i}\cdots\widehat{j}})_{IJ} \ \left(\text{or } \xi^{I'}\xi^{J'}(\sigma^{\widehat{i}\cdots\widehat{j}})_{I'J'}\right) \ \ (n \text{ is even}) \tag{64}$$

with a different number of indices $\widehat{i}\cdots\widehat{j}$, taken together, defines the h-spinor $\xi^{\underline{i}}$ to an accuracy to the sign. We emphasize that it is necessary to choose only those h-components of d-tensors (63) (or (64)) which are symmetric on pairs of indices $\underline{\alpha\beta}$ (or IJ (or $I'J'$)) and the number q of indices $\widehat{i}\cdots\widehat{j}$ satisfies the condition (as a respective consequence of the properties (60) and/or (61), (62))

$$n - 2q \equiv 0,1,7 \ (mod\ 8). \tag{65}$$

Of special interest is the case when

$$q = \frac{1}{2}(n \pm 1) \ \ (n \text{ is odd}) \tag{66}$$

or

$$q = \frac{1}{2}n \ \ (n \text{ is even}). \tag{67}$$

If all expressions (63) and/or (64) are zero for all values of q with the exception of one or two ones defined by the conditions (65), (66) (or (67)), the value $\xi^{\widehat{i}}$ (or ξ^{I} $(\xi^{I'})$) is called a fundamental h-spinor. Defining in a similar manner the fundamental v-spinors we can introduce fundamental d-spinors as pairs of fundamental h- and v-spinors. Here we remark that a h(v$_p$)-spinor $\xi^{\widehat{i}}$ $(\xi^{\widehat{a}_p})$ (we can also consider reduced components) is always a fundamental one for $n(m) < 7$, which is a consequence of (67)).

Finally, in this subsection, we note that the geometry of fundamental h- and v-spinors is similar to that of usual fundamental spinors (see Appendix to the monograph [28]). We omit such details in this work, but emphasize that constructions with fundamental d-spinors, for a la-space, must be adapted to the corresponding global splitting by N-connection of the space.

7. The differential geometry of D–spinors

This subsection is devoted to the differential geometry of d–spinors in higher order anisotropic spaces. We shall use denotations of type

$$v^{<\alpha>} = (v^i, v^{<a>}) \in \sigma^{<\alpha>} = (\sigma^i, \sigma^{<a>})$$

and

$$\zeta^{\underline{\alpha}_p} = (\zeta^{\underline{i}_p}, \zeta^{\underline{a}_p}) \in \sigma^{\alpha_p} = (\sigma^{i_p}, \sigma^{a_p})$$

for, respectively, elements of modules of d-vector and irreduced d-spinor fields (see details in [34]). D-tensors and d-spinor tensors (irreduced or reduced) will be interpreted as elements of corresponding σ–modules, for instance,

$$q^{<\alpha>}_{\ <\beta>...} \in \sigma^{<\alpha>}_{\ \ <\beta>...}, \psi^{\underline{\alpha}_p \quad \underline{\gamma}_p}_{\ \ \underline{\beta}_p \quad ...} \in \sigma^{\underline{\alpha}_p \quad \underline{\gamma}_p}_{\ \ \underline{\beta_p} \quad ...}, \xi^{I_p I'_p}_{\ \ J_p K'_p N'_p} \in \sigma^{I_p I'_p}_{\ \ J_p K'_p N'_p}, \ldots$$

We can establish a correspondence between the la-adapted metric $g_{\alpha\beta}$ (12) and d-spinor metric $\epsilon_{\underline{\alpha\beta}}$ (ϵ-objects (50) for both h- and v$_p$-subspaces of $\mathcal{E}^{<z>}$) of a ha-space $\mathcal{E}^{<z>}$ by using the relation

$$g_{<\alpha><\beta>} = -\frac{1}{N(n) + N(m_1) + \cdots + N(m_z)} \times \tag{68}$$

$$((\sigma_{(<\alpha>}(u))^{\underline{\alpha\beta}}(\sigma_{<\beta>)}(u))^{\underline{\delta\gamma}})\epsilon_{\underline{\alpha\gamma}}\epsilon_{\underline{\beta\delta}},$$

where

$$(\sigma_{<\alpha>}(u))^{\underline{\nu\gamma}} = l^{<\widehat{\alpha}>}_{<\alpha>}(u)(\sigma_{<\widehat{\alpha}>})^{<\underline{\nu}><\underline{\gamma}>}, \tag{69}$$

which is a consequence of formulas (43)-(50). In brief we can write (68) as

$$g_{<\alpha><\beta>} = \epsilon_{\underline{\alpha}_1\underline{\alpha}_2}\epsilon_{\underline{\beta}_1\underline{\beta}_2} \tag{70}$$

if the σ-objects are considered as a fixed structure, whereas ϵ-objects are treated as caring the metric "dynamics " , on la-space. This variant is used, for instance, in the so-called 2-spinor geometry [27, 28] and should be preferred if we have to make explicit the algebraic symmetry properties of d-spinor objects by using metric decomposition

(70). An alternative way is to consider as fixed the algebraic structure of ϵ-objects and to use variable components of σ-objects of type (69) for developing a variational d-spinor approach to gravitational and matter field interactions on ha-spaces (the spinor Ashtekar variables [7] are introduced in this manner).

We note that a d–spinor metric

$$\epsilon_{\underline{\nu\tau}} = \begin{pmatrix} \epsilon_{\underline{ij}} & 0 & \cdots & 0 \\ 0 & \epsilon_{\underline{a}_1\underline{b}_1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \epsilon_{\underline{a}_z\underline{b}_z} \end{pmatrix}$$

on the d-spinor space $\mathcal{S} = (\mathcal{S}_{(h)}, \mathcal{S}_{(v_1)}, \ldots, \mathcal{S}_{(v_z)})$ can have symmetric or antisymmetric h (v_p) -components $\epsilon_{\underline{ij}}$ ($\epsilon_{\underline{a}_p\underline{b}_p}$) , see ϵ-objects (50). For simplicity, in order to avoid cumbersome calculations connected with eight-fold periodicity on dimensions n and m_p of a ha-space $\mathcal{E}^{<z>}$, we shall develop a general d-spinor formalism only by using irreduced spinor spaces $\mathcal{S}_{(h)}$ and $\mathcal{S}_{(v_p)}$.

7.1. D-covariant derivation on ha–spaces

Let $\mathcal{E}^{<z>}$ be a ha-space. We define the action on a d-spinor of a d-covariant operator

$$\nabla_{<\alpha>} = (\nabla_i, \nabla_{<a>}) = (\sigma_{<\alpha>})^{\underline{\alpha}_1\underline{\alpha}_2}\nabla_{\underline{\alpha}_1\underline{\alpha}_2} = \left((\sigma_i)^{\underline{i}_1\underline{i}_2}\nabla_{\underline{i}_1\underline{i}_2},\ (\sigma_{<a>})^{\underline{a}_1\underline{a}_2}\nabla_{\underline{a}_1\underline{a}_2}\right)$$

$$= \left((\sigma_i)^{\underline{i}_1\underline{i}_2}\nabla_{\underline{i}_1\underline{i}_2},\ (\sigma_{a_1})^{\underline{a}_1\underline{a}_2}\nabla_{(1)\underline{a}_1\underline{a}_2}, \ldots, (\sigma_{a_p})^{\underline{a}_1\underline{a}_2}\nabla_{(p)\underline{a}_1\underline{a}_2}, \ldots, (\sigma_{a_z})^{\underline{a}_1\underline{a}_2}\nabla_{(z)\underline{a}_1\underline{a}_2}\right)$$

(in brief, we shall write

$$\nabla_{<\alpha>} = \nabla_{\underline{\alpha}_1\underline{\alpha}_2} = \left(\nabla_{\underline{i}_1\underline{i}_2},\ \nabla_{(1)\underline{a}_1\underline{a}_2}, \ldots, \nabla_{(p)\underline{a}_1\underline{a}_2}, \ldots, \nabla_{(z)\underline{a}_1\underline{a}_2}\right))$$

as maps

$$\nabla_{\underline{\alpha}_1\underline{\alpha}_2}\ :\ \sigma^{\underline{\beta}} \to \sigma^{\underline{\beta}}_{<\alpha>} = \sigma^{\underline{\beta}}_{\underline{\alpha}_1\underline{\alpha}_2} =$$

$$\left(\sigma^{\underline{\beta}}_{i} = \sigma^{\underline{\beta}}_{\underline{i}_1\underline{i}_2}, \sigma^{\underline{\beta}}_{(1)a_1} = \sigma^{\underline{\beta}}_{(1)\underline{\alpha}_1\underline{\alpha}_2}, \ldots, \sigma^{\underline{\beta}}_{(p)a_p} = \sigma^{\underline{\beta}}_{(p)\underline{\alpha}_1\underline{\alpha}_2}, \ldots, \sigma^{\underline{\beta}}_{(z)a_z} = \sigma^{\underline{\beta}}_{(z)\underline{\alpha}_1\underline{\alpha}_2}\right)$$

satisfying conditions

$$\nabla_{<\alpha>}(\xi^{\underline{\beta}} + \eta^{\underline{\beta}}) = \nabla_{<\alpha>}\xi^{\underline{\beta}} + \nabla_{<\alpha>}\eta^{\underline{\beta}},$$

and

$$\nabla_{<\alpha>}(f\xi^{\underline{\beta}}) = f\nabla_{<\alpha>}\xi^{\underline{\beta}} + \xi^{\underline{\beta}}\nabla_{<\alpha>}f$$

for every $\xi^{\underline{\beta}}, \eta^{\underline{\beta}} \in \sigma^{\underline{\beta}}$ and f being a scalar field on $\mathcal{E}^{<z>}$. It is also required that one holds the Leibnitz rule

$$(\nabla_{<\alpha>}\zeta_{\underline{\beta}})\eta^{\underline{\beta}} = \nabla_{<\alpha>}(\zeta_{\underline{\beta}}\eta^{\underline{\beta}}) - \zeta_{\underline{\beta}}\nabla_{<\alpha>}\eta^{\underline{\beta}}$$

and that $\nabla_{<\alpha>}$ is a real operator, i.e. it commuters with the operation of complex conjugation:

$$\overline{\nabla_{<\alpha>}\psi_{\underline{\alpha}\underline{\beta}\underline{\gamma}\ldots}} = \nabla_{<\alpha>}(\overline{\psi}_{\underline{\alpha}\underline{\beta}\underline{\gamma}\ldots}).$$

Let now analyze the question on uniqueness of action on d-spinors of an operator $\nabla_{<\alpha>}$ satisfying necessary conditions . Denoting by $\nabla^{(1)}_{<\alpha>}$ and $\nabla_{<\alpha>}$ two such d-covariant operators we consider the map

$$(\nabla^{(1)}_{<\alpha>} - \nabla_{<\alpha>}) : \sigma^{\underline{\beta}} \to \sigma^{\underline{\beta}}_{\underline{\alpha}_{\infty}\underline{\alpha}_{\in}}. \tag{71}$$

Because the action on a scalar f of both operators $\nabla^{(1)}_{\alpha}$ and ∇_{α} must be identical, i.e.

$$\nabla^{(1)}_{<\alpha>} f = \nabla_{<\alpha>} f,$$

the action (71) on $f = \omega_{\underline{\beta}}\xi^{\underline{\beta}}$ must be written as

$$(\nabla^{(1)}_{<\alpha>} - \nabla_{<\alpha>})(\omega_{\underline{\beta}}\xi^{\underline{\beta}}) = 0.$$

In consequence we conclude that there is an element $\Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}} \in \sigma_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}}$ for which

$$\nabla^{(1)}_{\underline{\alpha}_1\underline{\alpha}_2}\xi^{\underline{\gamma}} = \nabla_{\underline{\alpha}_1\underline{\alpha}_2}\xi^{\underline{\gamma}} + \Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}}\xi^{\underline{\beta}} \tag{72}$$

and

$$\nabla^{(1)}_{\underline{\alpha}_1\underline{\alpha}_2}\omega_{\underline{\beta}} = \nabla_{\underline{\alpha}_1\underline{\alpha}_2}\omega_{\underline{\beta}} - \Theta_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}}{}^{\underline{\gamma}}\omega_{\underline{\gamma}}\ .$$

The action of the operator (71) on a d-vector $v^{<\beta>} = v^{\underline{\beta}_1\underline{\beta}_2}$ can be written by using formula (72) for both indices $\underline{\beta}_1$ and $\underline{\beta}_2$:

$$(\nabla^{(1)}_{<\alpha>} - \nabla_{<\alpha>})v^{\underline{\beta}_1\underline{\beta}_2} = \Theta_{<\alpha>\underline{\gamma}}{}^{\underline{\beta}_1}v^{\underline{\gamma}\underline{\beta}_2} + \Theta_{<\alpha>\underline{\gamma}}{}^{\underline{\beta}_2}v^{\underline{\beta}_1\underline{\gamma}} =$$

$$(\Theta_{<\alpha>\underline{\gamma}_1}{}^{\underline{\beta}_1}\delta_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \Theta_{<\alpha>\underline{\gamma}_1}{}^{\underline{\beta}_2}\delta_{\underline{\gamma}_2}{}^{\underline{\beta}_1})v^{\underline{\gamma}_1\underline{\gamma}_2} = Q^{<\beta>}_{\ \ <\alpha><\gamma>}v^{<\gamma>},$$

where

$$Q^{<\beta>}_{\ \ <\alpha><\gamma>} = Q^{\underline{\beta}_1\underline{\beta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\ \underline{\gamma}_1\underline{\gamma}_2} = \Theta_{<\alpha>\underline{\gamma}_1}{}^{\underline{\beta}_1}\delta_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \Theta_{<\alpha>\underline{\gamma}_1}{}^{\underline{\beta}_2}\delta_{\underline{\gamma}_2}{}^{\underline{\beta}_1}. \tag{73}$$

The d-commutator $\nabla_{[<\alpha>}\nabla_{<\beta>]}$ defines the d-torsion (see (23)-(25) and (30)). So, applying operators $\nabla^{(1)}_{[<\alpha>}\nabla^{(1)}_{<\beta>]}$ and $\nabla_{[<\alpha>}\nabla_{<\beta>]}$ on $f = \omega_{\underline{\beta}}\xi^{\underline{\beta}}$ we can write

$$T^{(1)<\gamma>}_{\quad\ <\alpha><\beta>} - T^{<\gamma>}_{\ \ <\alpha><\beta>} = Q^{<\gamma>}_{\ \ <\beta><\alpha>} - Q^{<\gamma>}_{\ \ <\alpha><\beta>}$$

with $Q^{<\gamma>}_{\ \ <\alpha><\beta>}$ from (73).

The action of operator $\nabla^{(1)}_{<\alpha>}$ on d-spinor tensors of type $\chi_{\underline{\alpha}_1\underline{\alpha}_2\underline{\alpha}_3\ldots}{}^{\underline{\beta}_1\underline{\beta}_2\ldots}$ must be constructed by using formula (72) for every upper index $\underline{\beta}_1\underline{\beta}_2\cdots$ and formula (73) for every lower index $\underline{\alpha}_1\underline{\alpha}_2\underline{\alpha}_3\cdots$.

7.2. Infeld–van der Waerden coefficients and d-connections

Let

$$\delta_{\underline{\alpha}}^{\ \ \alpha} = \left(\delta_{\underline{\mathbf{1}}}^{\ \ i}, \delta_{\underline{\mathbf{2}}}^{\ \ i}, \dots, \delta_{\underline{\mathbf{N(n)}}}^{\ \ i}, \delta_{\underline{\mathbf{1}}}^{\ \ a}, \delta_{\underline{\mathbf{2}}}^{\ \ a}, \dots, \delta_{\underline{\mathbf{N(m)}}}^{\ \ i}\right)$$

be a d–spinor basis. The dual to it basis is denoted as

$$\delta_{\underline{\alpha}}^{\ \ \alpha} = \left(\delta_{\underline{i}}^{\ \ \mathbf{1}}, \delta_{\underline{i}}^{\ \ \mathbf{2}}, \dots, \delta_{\underline{i}}^{\ \ \underline{\mathbf{N(n)}}}, \delta_{\underline{i}}^{\ \ \mathbf{1}}, \delta_{\underline{i}}^{\ \ \mathbf{2}}, \dots, \delta_{\underline{i}}^{\ \ \underline{\mathbf{N(m)}}}\right).$$

A d-spinor $\kappa^{\underline{\alpha}} \in \sigma^{\ \underline{\alpha}}$ has components $\kappa^{\underline{\alpha}} = \kappa^{\underline{\alpha}}\delta_{\underline{\alpha}}^{\ \ \underline{\alpha}}$. Taking into account that

$$\delta_{\underline{\alpha}}^{\ \ \alpha}\delta_{\underline{\beta}}^{\ \ \beta}\nabla_{\underline{\alpha\beta}} = \nabla_{\underline{\alpha\beta}},$$

we write out the components $\nabla_{\underline{\alpha\beta}}\ \kappa^{\underline{\gamma}}$ as

$$\delta_{\underline{\alpha}}^{\ \ \alpha}\ \delta_{\underline{\beta}}^{\ \ \beta}\ \delta_{\underline{\gamma}}^{\ \ \gamma}\ \nabla_{\underline{\alpha\beta}}\kappa^{\underline{\gamma}} = \tag{74}$$

$$\delta_{\underline{\epsilon}}^{\ \ \tau}\ \delta_{\underline{\tau}}^{\ \ \gamma}\ \nabla_{\underline{\alpha\beta}}\kappa^{\underline{\epsilon}} + \kappa^{\underline{\epsilon}}\ \delta_{\underline{\epsilon}}^{\ \ \gamma}\ \nabla_{\underline{\alpha\beta}}\delta_{\underline{\epsilon}}^{\ \ \epsilon} = \nabla_{\underline{\alpha\beta}}\kappa^{\underline{\gamma}} + \kappa^{\underline{\epsilon}}\gamma^{\underline{\gamma}}_{\ \underline{\alpha\beta\epsilon}},$$

where the coordinate components of the d–spinor connection $\gamma^{\underline{\gamma}}_{\ \underline{\alpha\beta\epsilon}}$ are defined as

$$\gamma^{\underline{\gamma}}_{\ \underline{\alpha\beta\epsilon}} \doteq \delta_{\underline{\tau}}^{\ \ \gamma}\ \nabla_{\underline{\alpha\beta}}\delta_{\underline{\epsilon}}^{\ \ \tau}. \tag{75}$$

We call the Infeld - van der Waerden d-symbols a set of σ-objects $(\sigma_{\alpha})^{\underline{\alpha\beta}}$ parametrized with respect to a coordinate d-spinor basis. Defining

$$\nabla_{<\alpha>} = (\sigma_{<\alpha>})^{\underline{\alpha\beta}}\ \nabla_{\underline{\alpha\beta}},$$

introducing denotations

$$\gamma^{\underline{\gamma}}_{\ <\alpha>\underline{\tau}} \doteq \gamma^{\underline{\gamma}}_{\ \underline{\alpha\beta\tau}}\ (\sigma_{<\alpha>})^{\underline{\alpha\beta}}$$

and using properties (74) we can write relations

$$l^{<\alpha>}_{<\alpha>}\ \delta_{\underline{\beta}}^{\ \ \beta}\ \nabla_{<\alpha>}\kappa^{\underline{\beta}} = \nabla_{<\alpha>}\kappa^{\underline{\beta}} + \kappa^{\underline{\delta}}\ \gamma^{\underline{\beta}}_{\ <\alpha>\underline{\delta}} \tag{76}$$

and

$$l^{<\alpha>}_{<\alpha>}\ \delta_{\underline{\beta}}^{\ \ \beta}\ \nabla_{<\alpha>}\ \mu_{\underline{\beta}} = \nabla_{<\alpha>}\ \mu_{\underline{\beta}} - \mu_{\underline{\delta}}\gamma^{\underline{\delta}}_{\ <\alpha>\underline{\beta}} \tag{77}$$

for d-covariant derivations $\nabla_{\underline{\alpha}}\kappa^{\underline{\beta}}$ and $\nabla_{\underline{\alpha}}\ \mu_{\underline{\beta}}$.

We can consider expressions similar to (76) and (77) for values having both types of d-spinor and d-tensor indices, for instance,

$$l^{<\alpha>}_{<\alpha>}\ l^{<\gamma>}_{<\gamma>}\ \delta_{\underline{\delta}}^{\ \ \delta}\ \nabla_{<\alpha>}\theta_{\underline{\delta}}^{\ <\gamma>} = \nabla_{<\alpha>}\theta_{\underline{\delta}}^{\ <\gamma>} - \theta_{\underline{\epsilon}}^{\ <\gamma>}\gamma^{\underline{\epsilon}}_{\ <\alpha>\underline{\delta}} + \theta_{\underline{\delta}}^{\ <\tau>}\ \Gamma^{\ \ <\gamma>}_{<\alpha><\tau>}$$

(we can prove this by a straightforward calculation).

Now we shall consider some possible relations between components of d-connections $\gamma^{\underline{\epsilon}}_{\ <\alpha>\underline{\delta}}$ and $\Gamma^{\ <\gamma>}_{\ <\alpha><\tau>}$ and derivations of $(\sigma_{<\alpha>})^{\underline{\alpha\beta}}$. According to definitions (12) we can write

$$\Gamma^{<\alpha>}_{\ <\beta><\gamma>} = l^{<\alpha>}_{<\alpha>} \nabla_{<\gamma>} l^{<\alpha>}_{<\beta>} =$$

$$l^{<\alpha>}_{<\alpha>} \nabla_{<\gamma>} (\sigma_{<\beta>})^{\underline{\epsilon\tau}} = l^{<\alpha>}_{<\alpha>} \nabla_{<\gamma>} ((\sigma_{<\beta>})^{\underline{\epsilon\tau}} \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} \delta_{\underline{\tau}}^{\ \underline{\tau}}) =$$

$$l^{<\alpha>}_{<\alpha>} \delta_{\underline{\alpha}}^{\ \underline{\alpha}} \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} \nabla_{<\gamma>} (\sigma_{<\beta>})^{\underline{\alpha\epsilon}} + l^{<\alpha>}_{<\alpha>} (\sigma_{<\beta>})^{\underline{\epsilon\tau}} (\delta_{\underline{\tau}}^{\ \underline{\tau}} \nabla_{<\gamma>} \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} + \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} \nabla_{<\gamma>} \delta_{\underline{\tau}}^{\ \underline{\tau}}) =$$

$$l^{<\alpha>}_{\underline{\epsilon\tau}} \ \nabla_{<\gamma>} (\sigma_{<\beta>})^{\underline{\epsilon\tau}} + l^{<\alpha>}_{\underline{\mu\nu}} \delta_{\underline{\epsilon}}^{\ \underline{\mu}} \delta_{\underline{\tau}}^{\ \underline{\nu}} (\sigma_{<\beta>})^{\underline{\epsilon\tau}} (\delta_{\underline{\tau}}^{\ \underline{\tau}} \nabla_{<\gamma>} \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} + \delta_{\underline{\epsilon}}^{\ \underline{\epsilon}} \nabla_{<\gamma>} \delta_{\underline{\tau}}^{\ \underline{\tau}}),$$

where $l^{<\alpha>}_{<\alpha>} = (\sigma_{\underline{\epsilon\tau}})^{<\alpha>}$, from which it follows

$$(\sigma_{<\alpha>})^{\underline{\mu\nu}} (\sigma_{\underline{\alpha\beta}})^{<\beta>} \Gamma^{<\alpha>}_{\ <\gamma><\beta>} =$$

$$(\sigma_{\underline{\alpha\beta}})^{<\beta>} \nabla_{<\gamma>} (\sigma_{<\alpha>})^{\underline{\mu\nu}} + \delta_{\underline{\beta}}^{\ \underline{\nu}} \gamma^{\underline{\mu}}_{\ <\gamma>\underline{\alpha}} + \delta_{\underline{\alpha}}^{\ \underline{\mu}} \gamma^{\underline{\nu}}_{\ <\gamma>\underline{\beta}}.$$

Connecting the last expression on $\underline{\beta}$ and $\underline{\nu}$ and using an orthonormalized d-spinor basis when $\gamma^{\underline{\beta}}_{\ <\gamma>\underline{\beta}} = 0$ (a consequence from (75)) we have

$$\gamma^{\underline{\mu}}_{\ <\gamma>\underline{\alpha}} = \frac{1}{N(n) + N(m_1) + \cdots + N(m_z)} \times$$

$$(\Gamma^{\underline{\mu\beta}}_{\ \ <\gamma> \ \underline{\alpha\beta}} - (\sigma_{\underline{\alpha\beta}})^{<\beta>} \nabla_{<\gamma>} (\sigma_{<\beta>})^{\underline{\mu\beta}}), \tag{78}$$

where

$$\Gamma^{\underline{\mu\beta}}_{\ \ <\gamma> \ \underline{\alpha\beta}} = (\sigma_{<\alpha>})^{\underline{\mu\beta}} (\sigma_{\underline{\alpha\beta}})^{\beta} \Gamma^{<\alpha>}_{\ <\gamma><\beta>}. \tag{79}$$

We also note here that, for instance, for the canonical (see (18) and (19)) and Berwald (see (20)) connections, Christoffel d-symbols (see (21)) we can express d-spinor connection (79) through corresponding locally adapted derivations of components of metric and N-connection by introducing corresponding coefficients instead of $\Gamma^{<\alpha>}_{\ <\gamma><\beta>}$ in (79) and than in (78).

7.3. D–spinors of ha–space curvature and torsion

The d-tensor indices of the commutator (29), $\Delta_{<\alpha><\beta>}$, can be transformed into d-spinor ones:

$$\Box_{\underline{\alpha\beta}} = (\sigma^{<\alpha><\beta>})_{\underline{\alpha\beta}} \Delta_{\alpha\beta} = (\Box_{\underline{ij}}, \Box_{\underline{ab}}) = \tag{80}$$

$$(\Box_{\underline{ij}}, \Box_{\underline{a_1 b_1}}, \ldots, \Box_{\underline{a_p b_p}}, \ldots, \Box_{\underline{a_z b_z}}),$$

with h- and v_p-components,

$$\Box_{\underline{ij}} = (\sigma^{<\alpha><\beta>})_{\underline{ij}} \Delta_{<\alpha><\beta>} \text{ and } \Box_{\underline{ab}} = (\sigma^{<\alpha><\beta>})_{\underline{ab}} \Delta_{<\alpha><\beta>},$$

being symmetric or antisymmetric in dependence of corresponding values of dimensions n and m_p (see eight-fold parametrizations (50) and (51)). Considering the actions of operator (80) on d-spinors $\pi^{\underline{\gamma}}$ and $\mu_{\underline{\gamma}}$ we introduce the d-spinor curvature $X_{\underline{\delta}} \ {}^{\underline{\gamma}}{}_{\underline{\alpha\beta}}$ as to satisfy equations

$$\square_{\underline{\alpha\beta}}\ \pi^{\underline{\gamma}} = X_{\underline{\delta}} \ {}^{\underline{\gamma}}{}_{\underline{\alpha\beta}} \pi^{\underline{\delta}} \tag{81}$$

and

$$\square_{\underline{\alpha\beta}}\ \mu_{\underline{\gamma}} = X_{\underline{\gamma}} \ {}^{\underline{\delta}}{}_{\underline{\alpha\beta}} \mu_{\underline{\delta}}.$$

The gravitational d-spinor $\Psi_{\underline{\alpha\beta\gamma\delta}}$ is defined by a corresponding symmetrization of d-spinor indices:

$$\Psi_{\underline{\alpha\beta\gamma\delta}} = X_{(\underline{\alpha}|\underline{\beta}|\underline{\gamma\delta})}.$$

We note that d-spinor tensors $X_{\underline{\delta}} \ {}^{\underline{\gamma}}{}_{\underline{\alpha\beta}}$ and $\Psi_{\underline{\alpha\beta\gamma\delta}}$ are transformed into similar 2-spinor objects on locally isotropic spaces [27, 28] if we consider vanishing of the N-connection structure and a limit to a locally isotropic space.

Putting $\delta_{\underline{\gamma}} \ {}^{\underline{\gamma}}$ instead of $\mu_{\underline{\gamma}}$ in (81) we can express respectively the curvature and gravitational d-spinors as

$$X_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}} \square_{\underline{\alpha\beta}} \delta_{\underline{\gamma}} \ {}^{\underline{\tau}}$$

and

$$\Psi_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}} \square_{(\underline{\alpha\beta}} \delta_{\underline{\gamma})} \ {}^{\underline{\tau}}.$$

The d-spinor torsion $T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}$ is defined similarly as for d-tensors (see (30)) by using the d-spinor commutator (80) and equations

$$\square_{\underline{\alpha\beta}} f = T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}} \nabla_{\underline{\gamma}_1\underline{\gamma}_2} f. \tag{82}$$

The d-spinor components $R_{\underline{\gamma}_1\underline{\gamma}_2} \ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}$ of the curvature d-tensor $R_{\gamma} \ {}^{\delta}{}_{\alpha\beta}$ can be computed by using relations (79), and (80) and (82) as to satisfy the equations (the d-spinor analogous of equations (31))

$$(\square_{\underline{\alpha\beta}} - T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}} \nabla_{\underline{\gamma}_1\underline{\gamma}_2}) V^{\underline{\delta}_1\underline{\delta}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2} \ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}} V^{\underline{\gamma}_1\underline{\gamma}_2},$$

here d-vector $V^{\underline{\gamma}_1\underline{\gamma}_2}$ is considered as a product of d-spinors, i.e. $V^{\underline{\gamma}_1\underline{\gamma}_2} = \nu^{\underline{\gamma}_1}\mu^{\underline{\gamma}_2}$. We find

$$R_{\underline{\gamma}_1\underline{\gamma}_2} \ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}} = \left(X_{\underline{\gamma}_1} \ {}^{\underline{\delta}_1}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}} \ \ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} \right) \delta_{\underline{\gamma}_2} \ {}^{\underline{\delta}_2} + \tag{83}$$

$$\left(X_{\underline{\gamma}_2} \ {}^{\underline{\delta}_2}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}} \ \ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} \right) \delta_{\underline{\gamma}_1} \ {}^{\underline{\delta}_1}.$$

It is convenient to use this d-spinor expression for the curvature d-tensor

$$R_{\underline{\gamma}_1\underline{\gamma}_2} \ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} = \left(X_{\underline{\gamma}_1} \ {}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} \ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} \right) \delta_{\underline{\gamma}_2} \ {}^{\underline{\delta}_2} +$$

$$\left(X_{\underline{\gamma}_2} \ {}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} \ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} \right) \delta_{\underline{\gamma}_1} \ {}^{\underline{\delta}_1}$$

in order to get the d–spinor components of the Ricci d-tensor

$$R_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2}{}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\delta}_2} = \tag{84}$$

$$X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} + X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}$$

and this d-spinor decomposition of the scalar curvature:

$$q\overleftarrow{R} = R^{\underline{\alpha}_1\underline{\alpha}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2} = X^{\underline{\alpha}_1\underline{\delta}_1}{}_{\underline{\alpha}_1}{}^{\underline{\alpha}_2}{}_{\underline{\delta}_1\underline{\alpha}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_1}{}^{\underline{\alpha}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1} + \tag{85}$$

$$X^{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}^{\underline{\alpha}_2\underline{\alpha}_1}_{\underline{\alpha}_1}{}_{\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_2}.$$

Putting (84) and (85) into (34) and, correspondingly, (35) we find the d–spinor components of the Einstein and $\Phi_{<\alpha><\beta>}$ d–tensors:

$$\overleftarrow{G}_{<\gamma><\alpha>} = \overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} + \tag{86}$$

$$X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} -$$

$$\frac{1}{2}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\,\gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} +$$

$$X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}^{\underline{\beta}_2\underline{\beta}_1}_{\underline{\beta}_1}{}_{\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}]$$

and

$$\Phi_{<\gamma><\alpha>} = \Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = \frac{1}{2(n+m_1+\cdots+m_z)}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} + \tag{87}$$

$$T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\,\gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} + X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}^{\underline{\beta}_2\underline{\beta}_1}_{\underline{\beta}_1}{}_{\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}] -$$

$$\frac{1}{2}[X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} +$$

$$X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}].$$

The components of the conformal Weyl d-spinor can be computed by putting d-spinor values of the curvature (83) and Ricci (84) d-tensors into corresponding expression for the d-tensor (33). We omit this calculus in this work.

8. Field equations on ha-spaces

The problem of formulation gravitational and gauge field equations on different types of la-spaces is considered, for instance, in [23, 9, 6] and [42]. In this subsection we shall introduce the basic field equations for gravitational and matter field la-interactions in a generalized form for generic higher order anisotropic spaces.

8.1. Locally anisotropic scalar field interactions

Let $\varphi(u) = (\varphi_1(u), \varphi_2(u), \ldots, \varphi_k(u))$ be a complex k-component scalar field of mass μ on ha-space $\mathcal{E}^{<z>}$. The d-covariant generalization of the conformally invariant (in the massless case) scalar field equation [27, 28] can be defined by using the d'Alambert locally anisotropic operator [4, 38] $\Box = D^{<\alpha>}D_{<\alpha>}$, where $D_{<\alpha>}$ is a d-covariant derivation on $\mathcal{E}^{<z>}$ satisfying conditions (14) and (15) and constructed, for simplicity, by using Christoffel d–symbols (21) (all formulas for field equations and conservation values can be deformed by using corresponding deformations d–tensors $P^{<\alpha>}_{<\beta><\gamma>}$ from the Christoffel d–symbols, or the canonical d–connection to a general d-connection into consideration):

$$(\Box + \frac{n_E - 2}{4(n_E - 1)}\overleftarrow{R} + \mu^2)\varphi(u) = 0, \tag{88}$$

where $n_E = n + m_1 + \cdots + m_z$.We must change d-covariant derivation $D_{<\alpha>}$ into ${}^{\diamond}D_{<\alpha>} = D_{<\alpha>} + ieA_{<\alpha>}$ and take into account the d-vector current

$$J^{(0)}_{<\alpha>}(u) = i((\overline{\varphi}(u) D_{<\alpha>}\varphi(u) - D_{<\alpha>}\overline{\varphi}(u))\varphi(u))$$

if interactions between locally anisotropic electromagnetic field (d-vector potential $A_{<\alpha>}$), where e is the electromagnetic constant, and charged scalar field φ are considered. The equations (88) are (locally adapted to the N-connection structure) Euler equations for the Lagrangian

$$\mathcal{L}^{(0)}(u) = \tag{89}$$

$$\sqrt{|g|}\left[g^{<\alpha><\beta>}\delta_{<\alpha>}\overline{\varphi}(u)\,\delta_{<\beta>}\varphi(u) - \left(\mu^2 + \frac{n_E - 2}{4(n_E - 1)}\right)\overline{\varphi}(u)\varphi(u)\right],$$

where $|g| = detg_{<\alpha><\beta>}$.

The locally adapted variations of the action with Lagrangian (89) on variables $\varphi(u)$ and $\overline{\varphi}(u)$ leads to the locally anisotropic generalization of the energy-momentum tensor,

$$E^{(0,can)}_{<\alpha><\beta>}(u) = \delta_{<\alpha>}\overline{\varphi}(u)\,\delta_{<\beta>}\varphi(u) + \tag{90}$$

$$\delta_{<\beta>}\overline{\varphi}(u)\,\delta_{<\alpha>}\varphi(u) - \frac{1}{\sqrt{|g|}}g_{<\alpha><\beta>}\mathcal{L}^{(0)}(u),$$

and a similar variation on the components of a d-metric (12) leads to a symmetric metric energy-momentum d-tensor,

$$E^{(0)}_{<\alpha><\beta>}(u) = E^{(0,can)}_{(<\alpha><\beta>)}(u) - \tag{91}$$

$$\frac{n_E - 2}{2(n_E - 1)}\left[R_{(<\alpha><\beta>)} + D_{(<\alpha>}D_{<\beta>)} - g_{<\alpha><\beta>}\Box\right]\overline{\varphi}(u)\varphi(u).$$

Here we note that we can obtain a nonsymmetric energy-momentum d-tensor if we firstly vary on $G_{<\alpha><\beta>}$ and than impose constraints of type (10) in order to have a compatibility with the N-connection structure. We also conclude that the existence of a N-connection in dv-bundle $\mathcal{E}^{<z>}$ results in a nonequivalence of energy-momentum d-tensors (90) and (91), nonsymmetry of the Ricci tensor (see (29)), nonvanishing of the d-covariant derivation of the Einstein d-tensor (34), $D_{<\alpha>}\overleftarrow{G}^{<\alpha><\beta>} \neq 0$ and, in consequence, a corresponding breaking of conservation laws on ha-spaces when $D_{<\alpha>}E^{<\alpha><\beta>} \neq 0$. The problem of formulation of conservation laws on la-spaces is discussed in details and two variants of its solution (by using nearly autoparallel maps and tensor integral formalism on locally anisotropic and higher order multispaces) where proposed in [38]. In this subsection we shall present only straightforward generalizations of field equations and necessary formulas for energy-momentum d-tensors of matter fields on $\mathcal{E}^{<z>}$ considering that it is naturally that the conservation laws (usually being consequences of global, local and/or intrinsic symmetries of the fundamental space-time and of the type of field interactions) have to be broken on locally anisotropic spaces.

8.2. Proca equations on ha–spaces

Let consider a d-vector field $\varphi_{<\alpha>}(u)$ with mass μ^2 (locally anisotropic Proca field) interacting with exterior la-gravitational field. From the Lagrangian

$$\mathcal{L}^{(1)}(u) = \sqrt{|g|}\left[-\frac{1}{2}\overline{f}_{<\alpha><\beta>}(u)\, f^{<\alpha><\beta>}(u) + \mu^2\overline{\varphi}_{<\alpha>}(u)\,\varphi^{<\alpha>}(u)\right], \qquad (92)$$

where $f_{<\alpha><\beta>} = D_{<\alpha>}\varphi_{<\beta>} - D_{<\beta>}\varphi_{<\alpha>}$, one follows the Proca equations on higher order anisotropic spaces

$$D_{<\alpha>}f^{<\alpha><\beta>}(u) + \mu^2\varphi^{<\beta>}(u) = 0. \qquad (93)$$

Equations (93) are a first type constraints for $\beta = 0$. Acting with $D_{<\alpha>}$ on (93), for $\mu \neq 0$ we obtain second type constraints

$$D_{<\alpha>}\varphi^{<\alpha>}(u) = 0. \qquad (94)$$

Putting (94) into (93) we obtain second order field equations with respect to $\varphi_{<\alpha>}$:

$$\Box\varphi_{<\alpha>}(u) + R_{<\alpha><\beta>}\varphi^{<\beta>}(u) + \mu^2\varphi_{<\alpha>}(u) = 0. \qquad (95)$$

The energy-momentum d-tensor and d-vector current following from the (95) can be written as

$$E^{(1)}_{<\alpha><\beta>}(u) = -g^{<\varepsilon><\tau>}\left(\overline{f}_{<\beta><\tau>}f_{<\alpha><\varepsilon>} + \overline{f}_{<\alpha><\varepsilon>}f_{<\beta><\tau>}\right) +$$
$$\mu^2\left(\overline{\varphi}_{<\alpha>}\varphi_{<\beta>} + \overline{\varphi}_{<\beta>}\varphi_{<\alpha>}\right) - \frac{g_{<\alpha><\beta>}}{\sqrt{|g|}}\mathcal{L}^{(1)}(u).$$

and

$$J^{(1)}_{<\alpha>}(u) = i\left(\overline{f}_{<\alpha><\beta>}(u)\,\varphi^{<\beta>}(u) - \overline{\varphi}^{<\beta>}(u)\, f_{<\alpha><\beta>}(u)\right).$$

For $\mu = 0$ the d-tensor $f_{<\alpha><\beta>}$ and the Lagrangian (92) are invariant with respect to locally anisotropic gauge transforms of type

$$\varphi_{<\alpha>}(u) \rightarrow \varphi_{<\alpha>}(u) + \delta_{<\alpha>}\Lambda(u),$$

where $\Lambda(u)$ is a d-differentiable scalar function, and we obtain a locally anisotropic variant of Maxwell theory.

8.3. Higher order anisotropic gravitons

Let a massless d-tensor field $h_{<\alpha><\beta>}(u)$ is interpreted as a small perturbation of the locally anisotropic background metric d-field $g_{<\alpha><\beta>}(u)$. Considering, for simplicity, a torsionless background we have locally anisotropic Fierz–Pauli equations

$$\Box h_{<\alpha><\beta>}(u) + 2R_{<\tau><\alpha><\beta><\nu>}(u)\ h^{<\tau><\nu>}(u) = 0$$

and d–gauge conditions

$$D_{<\alpha>}h^{<\alpha>}_{<\beta>}(u) = 0, \quad h(u) \equiv h^{<\alpha>}_{<\beta>}(u) = 0,$$

where $R_{<\tau><\alpha><\beta><\nu>}(u)$ is curvature d-tensor of the la-background space (these formulae can be obtained by using a perturbation formalism with respect to $h_{<\alpha><\beta>}(u)$ developed in [16]; in our case we must take into account the distinguishing of geometrical objects and operators on ha–spaces).

8.4. Higher order anisotropic Dirac equations

Let denote the Dirac d–spinor field on $\mathcal{E}^{<z>}$ as $\psi(u) = (\psi^{\underline{\alpha}}(u))$ and consider as the generalized Lorentz transforms the group of automorphysm of the metric $G_{<\widehat{\alpha}><\widehat{\beta}>}$ (see the ha-frame decomposition of d-metric (12), (68) and (69)).The d-covariant derivation of field $\psi(u)$ is written as

$$\overrightarrow{\nabla_{<\alpha>}}\psi = \left[\delta_{<\alpha>} + \frac{1}{4}C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}}(u)\ l^{\widehat{\alpha}}_{<\alpha>}(u)\,\sigma^{\widehat{\beta}}\sigma^{\widehat{\gamma}}\right]\psi, \tag{96}$$

where coefficients $C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}} = \left(D_{<\gamma>}l^{<\alpha>}_{\widehat{\alpha}}\right) l_{\widehat{\beta}<\alpha>}l^{<\gamma>}_{\widehat{\gamma}}$ generalize for ha-spaces the corresponding Ricci coefficients on Riemannian spaces [15]. Using σ-objects $\sigma^{<\alpha>}(u)$ (see (44) and (60)–(62)) we define the Dirac equations on ha–spaces:

$$(i\sigma^{<\alpha>}(u)\,\overrightarrow{\nabla_{<\alpha>}} - \mu)\psi = 0,$$

which are the Euler equations for the Lagrangian

$$\mathcal{L}^{(1/2)}(u) = \sqrt{|g|}\{[\psi^{+}(u)\,\sigma^{<\alpha>}(u)\,\overrightarrow{\nabla_{<\alpha>}}\psi(u) - \tag{97}$$

$$(\overrightarrow{\nabla_{<\alpha>}}\psi^{+}(u))\sigma^{<\alpha>}(u)\,\psi(u)] - \mu\psi^{+}(u)\,\psi(u)\},$$

where $\psi^+(u)$ is the complex conjugation and transposition of the column $\psi(u)$.

From (97) we obtain the d-metric energy-momentum d-tensor

$$E^{(1/2)}_{<\alpha><\beta>}(u) = \frac{i}{4}[\psi^+(u)\,\sigma_{<\alpha>}(u)\,\overrightarrow{\nabla_{<\beta>}}\psi(u) + \psi^+(u)\,\sigma_{<\beta>}(u)\,\overrightarrow{\nabla_{<\alpha>}}\psi(u) -$$

$$(\overrightarrow{\nabla_{<\alpha>}}\psi^+(u))\sigma_{<\beta>}(u)\,\psi(u) - (\overrightarrow{\nabla_{<\beta>}}\psi^+(u))\sigma_{<\alpha>}(u)\,\psi(u)]$$

and the d-vector source

$$J^{(1/2)}_{<\alpha>}(u) = \psi^+(u)\,\sigma_{<\alpha>}(u)\,\psi(u).$$

We emphasize that la-interactions with exterior gauge fields can be introduced by changing the higher order anisotropic partial derivation from (96) in this manner:

$$\delta_\alpha \to \delta_\alpha + ie^\star B_\alpha,$$

where $e^\star$ and B_α are respectively the constant d-vector potential of locally anisotropic gauge interactions on higher order anisotropic spaces (see [42]).

8.5. D–spinor locally anisotropic Yang–Mills fields

We consider a dv-bundle $\mathcal{B}_E,\ \pi_B : \mathcal{B} \to \mathcal{E}^{<z>}$, on ha-space $\mathcal{E}^{<z>}$. Additionally to d-tensor and d-spinor indices we shall use capital Greek letters, $\Phi, \Upsilon, \Xi, \Psi, \ldots$ for fibre (of this bundle) indices (see details in [27, 28] for the case when the base space of the v-bundle π_B is a locally isotropic space-time). Let $\underline{\nabla}_{<\alpha>}$ be, for simplicity, a torsionless, linear connection in $\mathcal{B}_E$ satisfying conditions:

$$\underline{\nabla}_{<\alpha>} : \Upsilon^\Theta \to \Upsilon^\Theta_{<\alpha>} \quad \left[\text{or } \Xi^\Theta \to \Xi^\Theta_{<\alpha>}\right],$$

$$\underline{\nabla}_{<\alpha>}\left(\lambda^\Theta + \nu^\Theta\right) = \underline{\nabla}_{<\alpha>}\lambda^\Theta + \underline{\nabla}_{<\alpha>}\nu^\Theta,$$

$$\underline{\nabla}_{<\alpha>}(f\lambda^\Theta) = \lambda^\Theta\underline{\nabla}_{<\alpha>}f + f\underline{\nabla}_{<\alpha>}\lambda^\Theta, \quad f \in \Upsilon^\Theta\ [\text{or } \Xi^\Theta],$$

where by Υ^Θ $\left(\Xi^\Theta\right)$ we denote the module of sections of the real (complex) v-bundle $\mathcal{B}_E$ provided with the abstract index Θ. The curvature of connection $\underline{\nabla}_{<\alpha>}$ is defined as

$$K_{<\alpha><\beta>}{}^{\Theta}{}_{\Omega}\lambda^\Omega = \left(\underline{\nabla}_{<\alpha>}\underline{\nabla}_{<\beta>} - \underline{\nabla}_{<\beta>}\underline{\nabla}_{<\alpha>}\right)\lambda^\Theta.$$

For Yang-Mills fields as a rule one considers that $\mathcal{B}_E$ is enabled with a unitary (complex) structure (complex conjugation changes mutually the upper and lower Greek indices). It is useful to introduce instead of $K_{<\alpha><\beta>}{}^{\Theta}{}_{\Omega}$ a Hermitian matrix $F_{<\alpha><\beta>}{}^{\Theta}{}_{\Omega} = i\ K_{<\alpha><\beta>}{}^{\Theta}{}_{\Omega}$ connected with components of the Yang-Mills d-vector potential $B_{<\alpha>}{}^{\Phi}{}_{\Xi}$ according the formula:

$$\frac{1}{2}F_{<\alpha><\beta>}{}^{\Phi}{}_{\Xi} = \underline{\nabla}_{[<\alpha>}B_{<\beta>]}{}^{\Phi}{}_{\Xi} - iB_{[<\alpha>}{}^{\Phi}{}_{|\Lambda|}B_{<\beta>]}{}^{\Lambda}{}_{\Xi}, \tag{98}$$

where the la-space indices commute with capital Greek indices. The gauge transforms are written in the form:

$$B_{<\alpha>\Theta}^{\quad\Phi} \mapsto B_{<\alpha>\widehat{\Theta}}^{\quad\widehat{\Phi}} = B_{<\alpha>\Theta}^{\quad\Phi}\ s_{\Phi}^{\ \widehat{\Phi}}\ q_{\widehat{\Theta}}^{\ \Theta} + is_{\Theta}^{\ \widehat{\Phi}}\underline{\nabla}_{<\alpha>}\ q_{\widehat{\Theta}}^{\ \Theta},$$

$$F_{<\alpha><\beta>\Xi}^{\qquad\Phi} \mapsto F_{<\alpha><\beta>\widehat{\Xi}}^{\qquad\widehat{\Phi}} = F_{<\alpha><\beta>\Xi}^{\qquad\Phi} s_{\Phi}^{\ \widehat{\Phi}} q_{\widehat{\Xi}}^{\ \Xi},$$

where matrices $s_{\Phi}^{\ \widehat{\Phi}}$ and $q_{\widehat{\Xi}}^{\ \Xi}$ are mutually inverse (Hermitian conjugated in the unitary case). The Yang-Mills equations on torsionless la-spaces [42] are written in this form:

$$\underline{\nabla}^{<\alpha>} F_{<\alpha><\beta>\Theta}^{\qquad\Psi} = J_{<\beta>\ \Theta}^{\quad\ \Psi}, \tag{99}$$

$$\underline{\nabla}_{[<\alpha>} F_{<\beta><\gamma>]\Theta}^{\qquad\quad\Xi} = 0. \tag{100}$$

We must introduce deformations of connection of type $\underline{\nabla}_{\alpha}^{\star} \longrightarrow \underline{\nabla}_{\alpha} + P_{\alpha}$, (the deformation d-tensor P_{α} is induced by the torsion in dv-bundle $\mathcal{B}_E$) into the definition of the curvature of ha-gauge fields (98) and motion equations (99) and (100) if interactions are modeled on a generic ha-space.

8.6. D–spinor Einstein–Cartan equations

Now we can write out the field equations of the Einstein-Cartan theory in the d-spinor form. So, for the Einstein equations (34) we have

$$\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} + \lambda\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2} = \kappa E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

with $\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2}$ from (86), or, by using the d-tensor (87),

$$\Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} + (\frac{\overleftarrow{R}}{4} - \frac{\lambda}{2})\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2} = -\frac{\kappa}{2}E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

which are the d-spinor equivalent of the equations (35). These equations must be completed by the algebraic equations (36) for the d-torsion and d-spin density with d-tensor indices changed into corresponding d-spinor ones.

9. Summary and outlook

We have developed the spinor differential geometry of distinguished vector bundles provided with nonlinear and distinguished connections and metric structures and shown in detail the way of formulation the theory of fundamental field (gravitational, gauge and spinor) interactions on generic higher order anisotropic spaces.

We investigated the problem of definition of spinors on spaces with higher order anisotropy. Our approach is based on the formalism of Clifford distinguished (by a nonlinear connection structure) algebras. We introduced spinor structures on higher order anisotropic spaces as Clifford distinguished module structures on distinguished vector bundles. We also proposed the second definition, as distinguished spinor structures, by using Clifford fibrations. It was shown that almost Hermitian models of generalized Lagrange spaces, H^{2n}–spaces admit as a proper characteristic the almost complex spinor structures.

It should be noted that we introduced [34, 35, 43] distinguished spinor structures in an algebraic topological manner, and that in our considerations the compatibility of distinguished connection and metric, adapted to a given nonlinear connection, plays a crucial role. The Yano and Ishihara method of lifting of geometrical objects in the total spaces of tangent bundles [46] and the general formalism for vector bundles of Miron and Anastasiei [22, 23] and for higher order Lagrange spaces of Miron and Atanasiu [24] clearing up the possibility and way of definition of spinors on higher order anisotropic spaces. Even a straightforward definition of spinors on Finsler and Lagrange spaces, and, of course, on various theirs extensions, with general noncompatible connection and metric structures, is practically impossible (if spinors are introduced locally with respect to a given metric quadratic form, the spinor constructions will not be invariant on parallel transports), we can avoid this difficulty by lifting in a convenient manner the geometric objects and physical values from the base of a locally anisotropic space on the tangent bundles of vector and tangent bundles under consideration. We shall introduce corresponding discordance laws and values and define nonstandard spinor structures by using nonmetrical distinguished tensors (see such constructions for locally isotropic curved spaces with torsion and nonmetricity in [20]).

The distinguishing by a nonlinear connection structure of the multidimensional space into horizontal and vertical subbundles points out to the necessity to start up the spinor constructions for locally anisotropic spaces with a study of distinguished Clifford algebras for vector spaces split into horizontal and vector subspaces. The distinguished spinor objects exhibit a eight-fold periodicity on dimensions of the mentioned subspaces. As it was shown in [34, 35], see also Sections 3–5 of this work, a corresponding distinguished spinor technique can be developed, which is a generalization for higher dimensional with nonlinear connection structure of that proposed by Penrose and Rindler [26, 27, 28] for locally isotropic curved spaces, if the locally adapted to the nonlinear connection structures distinguished spinor and distinguished vector frames are used. It is clear the distinguished spinor calculus is more tedious than the two–spinor one for Einstein spaces because of multidimensional and multiconnection character of generic higher order anisotropic spaces. The distinguished spinor differential geometry formulated in Section 7 can be considered as a branch of the geometry of Clifford fibrations for vector bundles provided with nonlinear connection, distinguished connection and metric structures. We have emphasized only the features containing distinguished spinor torsions and curvatures which are necessary for a distinguished spinor formulation of locally anisotropic gravity. To develop a conformally invariant distinguished spinor calculus is possible only for a particular class of higher order anisotropic spaces when the Weyl distinguished tensor (33) is defined by the nonlinear connection and distinguished metric structures. In general, we have to extend the class of conformal transforms to that of nearly autoparallel maps of higher order anisotropic spaces (see [43, 44, 45, 39]).

Having fixed compatible nonlinear connection, distinguished connection and metric structures on a distinguished vector bundle $\mathcal{E}^{<z>}$ we can develop physical models on this space by using a covariant variational distinguished tensor calculus as on Riemann–Cartan spaces (really there are specific complexities because, in general, the Ricci distinguished tensor is not symmetric and the locally anisotropic frames are nonholonomic). The systems of basic field equations for fundamental matter (scalar, Proca and Dirac) fields and gauge and gravitational fields have been introduced in a geometric manner by using distinguished covariant operators and locally anisotropic frame decompositions of distinguished metric. These equations and expressions for energy–momentum distinguished tensors and distinguished vector currents can be established by using the standard variational procedure, but correspondingly adapted to the nonlinear connection structure if we work by using locally adapted bases.

Let us try to summarize our results, discuss their possible implications and make the basic conclusions. Firstly, we have shown that the Einstein–Cartan theory has a natural extension for a various class of higher order anisotropic spaces. Following the R. Miron, M. Anastasiei and Gh. Atanasiu approach [22, 23, 24] to the geometry of locally anisotropic and higher order anisotropic spaces it becomes evident the possibility and manner of formulation of classical and quantum field theories on such spaces. Here we note that in locally anisotropic theories we have an additional geometric structure, the nonlinear connection. From physical point of view it can be interpreted, for instance, as a fundamental field managing the dynamics of splitting of higher–dimensional space–time into the four–dimensional and compactified ones. We can also consider the nonlinear connection as a generalized type of gauge field which reflects some specifics of higher order anisotropic field interactions and possible intrinsic structures of higher order anisotropic spaces. It was convenient to analyze the geometric structure of different variants of higher order anisotropic spaces (for instance, Finsler, Lagrange and generalized Lagrange spaces) in order to make obvious physical properties and compare theirs perspectives in developing of new models of higher order anisotropic gravity, locally anisotropic strings and higher order anisotropic superstrings [37, 41].

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